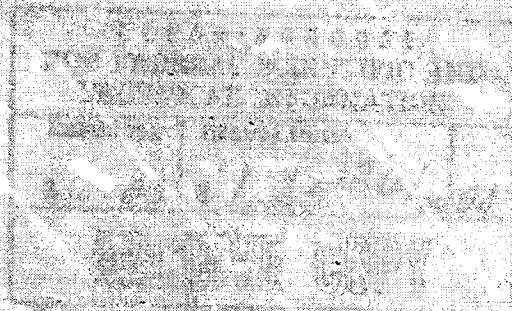


AD 633 351



**BEST  
AVAILABLE COPY**

# COLUMBIA UNIVERSITY

IN THE CITY OF NEW YORK

SCHOOL OF ENGINEERING AND APPLIED SCIENCE  
ELECTRONICS RESEARCH LABORATORIES

63<sup>rd</sup> WEST 125th STREET  
NEW YORK, NEW YORK 10027

CU-28-65-AF-1478-ERL

September 30, 1965

TECHNICAL REPORT T-2/318

DETECTABILITY OF RADAR ECHOES IN  
NOISE AND CLUTTER INTERFERENCE

By

C. Pedersen

Prepared for

Director

Advanced Research Projects Agency  
Washington, D. C. 20301

and

Electronics Division  
Director of Engineering Sciences  
Air Force Office of Scientific Research  
Office of Aerospace Research  
U. S. Air Force  
Washington, D. C. 20333

Contract No. AF 49(638)-1478  
ARPA Order No. 279



**BLANK PAGE**



ABSTRACT

The radar echo from a moving, non-scintillating point target is assumed to be received in the presence of noise and clutter interference. For best detection of the echo, the radar receiver will include a data processor which maximizes the signal-to-interference power ratio at its output.

This report presents the integral equation which determines the weighting function of the optimum data processor. Several general forms of solution are also presented, along with specialized forms of solution which are applicable for appropriately simplified sources of clutter. These solutions show that the optimum processor has no simple relation to either the transmitted waveform or to the clutter dispersion function but, rather is markedly influenced by the relative power levels of the noise and clutter components of the interference. Ultimately, the extent to which clutter can be rejected by a suitably designed signal processor will be limited by the (often neglected) noise level which accompanies the clutter.

An upper bound is derived for the decibel difference between signal-to-interference ratios which may exist at the outputs of an optimum processor and conventional "matched" processor, respectively, when identical signal and interference waveforms enter each processor. The bound is generally applicable and depends only upon a parameter related to clutter-to-noise ratio. It indicates that the greatest potentialities for performance improvement, through use of an optimum processor, exist only for large clutter-to-noise ratios.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Optimum processors and their performance are derived or computed for a number of particular cases involving different echo waveforms and sources of clutter. These results highlight many aspects of the problem of detecting radar echoes in noise and clutter interference.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## AUTHORIZATION


The research described in this report was performed at the Electronics Research Laboratories of the School of Engineering and Applied Science of Columbia University. This report was prepared by C. Pedersen.

This project is directed by the Advanced Research Projects Agency of the Department of Defense and is administered by the Air Force Office of Scientific Research under Contract AF 49(638)-1478.

Submitted by:

R. I. Bernstein  
Associate Director

Approved by:

  
for L. H. O'Neill  
Professor of Electrical Engineering  
Director

**BLANK PAGE**

TABLE OF CONTENTS

	<u>Page</u>
ABSTRACT	iii
I. INTRODUCTION	1
II. HISTORICAL BACKGROUND	8
A. DETECTION OF TIME- AND FREQUENCY- DISPERSED SIGNALS	9
B. PERFORMANCE OF MATCHED FILTER RECEIVERS IN CLUTTER	10
C. DESIGN OR DISCOVERY OF "GOOD" WAVEFORMS	11
D. DESIGN OF SYSTEMS FOR MOVING TARGET INDICATION (MTI)	12
E. PERFORMANCE OF OPTIMUM RECEIVERS IN CLUTTER	12
III. RÉSUMÉ OF RESULTS AND CONCLUSIONS	15
A. PROBLEM FORMULATION	16
B. INTEGRAL EQUATION SOLUTIONS	18
C. PERFORMANCE IMPROVEMENT BOUND	23
D. PARTICULAR CASES	25
IV. PROBLEM FORMULATION	28
A. DESCRIPTION OF A RADAR SYSTEM	28
1. Transmitter	28
2. Receiver	30
3. Data Processor	30
B. THE NATURE OF CLUTTER	33
1. The Clutter Covariance Function	36
C. THE OUTPUT SIGNAL-TO-INTERFERENCE RATIO	39
D. THE OPTIMUM LINEAR PROCESSOR	42
1. Formal Solution for the General Case	43
2. Solution for Uniformly Extended Clutter	45

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## TABLE OF CONTENTS (CONT'D.)

	<u>Page</u>
V.        AMBIGUITY FUNCTIONS	49
A.    A GENERAL AMBIGUITY FUNCTION	49
B.    UNIT TOTAL AMBIGUITY	52
C.    AMBIGUITY FUNCTIONS FOR STATIONARY SYSTEMS	55
VI.       GENERAL FORMS OF SOLUTION	56
A.    DISCUSSION OF THE INTEGRAL EQUATION	56
B.    SOLUTION IN TERMS OF EIGENFUNCTIONS	59
1.    The Schmidt - Hilbert Solution	63
2.    Bounds for Eigenvalues	68
C.    SOLUTION AS A RATIONAL FUNCTION OF CLUTTER-TO-NOISE RATIO	70
1.    The Fredholm Solution	71
D.    SOLUTION IN TERMS OF ECHO WAVEFORMS	74
VII.      ASYMPTOTIC FORMS OF SOLUTION	79
A.    SOLUTION WHEN NOISE INTERFERENCE IS DOMINANT	79
1.    Comparison with Matched Filter	82
B.    SOLUTION WHEN NOISE INTERFERENCE IS NEGLIGIBLE	83
C.    A HYPOTHETICAL SOLUTION	89
VIII.     A MODE OF SOLUTION FOR STATIONARY CLUTTER	91
A.    PRELIMINARY DISCUSSION	91
B.    LINEAR DIFFERENTIAL AND DIFFERENCE OPERATORS	93
C.    THE EQUIVALENT DIFFERENCE-DIFFERENTIAL EQUATION	96
D.    DISCUSSION OF THE SOLUTION	99

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## TABLE OF CONTENTS (CONT'D.)

	<u>Page</u>
IX. A GENERAL PERFORMANCE IMPROVEMENT BOUND	103
A. DERIVATION	103
B. DISCUSSION	107
X. GAUSSIAN ECHOES IN UNIFORMLY EXTENDED GAUSSIAN CLUTTER	111
A. ANALYTIC RESULTS	111
B. DISCUSSION OF ANALYTIC RESULTS	117
1. Signal-to-Clutter Ratio	117
2. Frequency Response Function	120
3. Ambiguity Function	126
C. NUMERICAL RESULTS (FOR CLUTTER PLUS NOISE)	132
1. Signal-to-Interference Ratio	132
2. Frequency Response Functions	144
D. PERFORMANCE IMPROVEMENT OVER THE SIMPLE MATCHED FILTER	148
XI. DETECTION OF GAUSSIAN PULSES IN CLUTTER WITH GAUSSIAN DELAY AND DOPPLER PROFILES	156
A. ANALYTIC RESULTS	156
B. DISCUSSION OF ANALYTIC RESULTS	160
1. Structure of the Optimum Processor	162
2. Signal-to-Interference Ratio	166
XII. DETECTION OF RECTANGULAR PULSES IN UNIFORMLY EXTENDED CLUTTER FROM A STATIONARY SOURCE	170
A. THE DIFFERENCE-DIFFERENTIAL EQUATION	170
B. A GENERAL SOLUTION	173
C. SMALL-NOISE SOLUTION	186
1. Discussion	197
D. SOLUTION FOR LARGE NOISE	200

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

TABLE OF CONTENTS (CONT'D.)

	<u>Page</u>
XIII. DETECTION OF RECTANGULAR PULSES IN CLUTTER FROM A STATIONARY SOURCE OF FINITE EXTENT	205
A. SOURCE WITH UNIFORM CROSS SECTION	207
1. A Set of Reciprocal Waveforms	208
2. Optimum Processor (No Noise)	213
B. SOURCE WITH NON-UNIFORM CROSS SECTION	218
APPENDICES	227
A. FORMAL EXPRESSIONS FOR AN OPTIMAL SYSTEM	227
B. ANALYSIS IN THE FREQUENCY DOMAIN	231
C. THE KANTOROVICH INEQUALITY	236
D. THE GAUSSIAN FUNCTION	240
E. APPROXIMATE INTEGRATION BY HERMITE-GAUSS QUADRATURE	241
REFERENCES	244



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## LIST OF FIGURES

<u>Figure</u>		<u>Page</u>
1	A Canonical Radar System	29
2	Conceivable Weight Functions for Non-Stationary Interference	32
3	Correlation in Clutter Returns	34
4	Variation of Performance Improvement Bound with Clutter-to-Noise-Ratio Parameter	108
5	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts	118
6	Frequency Response Functions for Detecting Gaussian Pulses with No Doppler Shift	121
7	Frequency Response Functions for Detecting Gaussian Pulses with Doppler Shift of $2W_q$ cps	122
8	Doppler Profiles of Ambiguity Functions for Detecting Gaussian Pulses with No Doppler Shift	128
9	Doppler Profiles of Ambiguity Functions for Detecting Gaussian Pulses with Doppler Shift of $2W_q$ cps	129
10a	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts and $R_c = \infty$	134
10b	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts and $R_c = 60$ db	135

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## LIST OF FIGURES (CONT'D.)

<u>Figure</u>		<u>Page</u>
10c	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts and $R_c = 50$ db	136
10d	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts and $R_c = 40$ db	137
10e	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts and $R_c = 30$ db	138
10f	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts and $R_c = 20$ db	139
10g	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts and $R_c = 10$ db	140
10h	$\mu_o$ as a Function of Signal Bandwidth for Various Doppler Frequency Shifts and $R_c = 0$ db	141
11	Frequency Response Functions for Detecting Gaussian Pulses with No Doppler Shift in Various Clutter Levels	145
12	Frequency Response Functions for Detecting Gaussian Pulses with Doppler Shift of $2W_q$ cps in Various Noise Levels	146
13	Illustrative Example of Maximum Performance Difference	150
14	Empirical Performance Improvement Data as a Function of Clutter-to-Noise Ratio	151

LIST OF FIGURES (CONT'D.)

<u>Figure</u>		<u>Page</u>
15	Optimum and Matched Processor Performance for Various Doppler Shifts, and $R_c = 60$ db	153
16	Empirical Performance Improvement Data as a Function of Echo Doppler Frequency	154
17	Optimum Processor for a Particular Sort of Non-Stationary Interference ( $N_0 = 0$ )	163
18	Range Delay Profile of Gaussian Pulse in Non-Stationary Gaussian Clutter	167
19	A Typical Quadruplet of Characteristic Roots	177
20	Functions Determining Locations of Characteristic Roots	181
21	Characteristic Root Loci for Small $N_0$ .	189
22a	Characteristic Roots for Small $N_0$ (First Quadrant; Large Scale)	192
22b	Characteristic Roots for Small $N_0$ (First Quadrant; Small Scale)	193
23	Optimum Zero-Noise Processor for a Rectan- gular Pulse in Uniformly Extended Clut- ter from a Stationary Source	198
24	Optimum Small-Noise Processor for a Rectan- gular Pulse in Uniformly Extended Clutter from a Stationary Source	199
25	Optimum Large-Noise Processors for a Rectan- gular Pulse in Uniformly Extended Clutter from a Stationary Source	203

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

LIST OF FIGURES (CONT'D.)

<u>Figure</u>		<u>Page</u>
26	A Set of Waveforms	209
27	A Reciprocal Waveform	211
28	Optimum Zero-Noise Processor for a Rectangular Pulse in Uniform Clutter of Finite Extent	215
29	Optimum Zero-Noise Processors for Various Pulse-Locations in Uniform Clutter of Finite Extent	217
30	Optimum Zero-Noise Processor for a Rectangular Pulse in Non-Uniform Clutter	225

## I. INTRODUCTION

The problem of devising radar systems which will operate effectively in an interference environment which includes both noise and clutter interference is a problem of considerable theoretical and practical interest. This dissertation considers one possible approach to the problem.

This dissertation presents the results of a theoretical inquiry into the properties of a class of signal processors which are optimum for detecting the presence of radar echoes in a mixture of noise and clutter interference. The reasons for considering this particular formulation of the problem will emerge from the following discussion.

A common goal of many radar systems which are either conjectured or actually constructed is to determine one or more components of the position or velocity, or both, of all targets within some field of view. Means for achieving this goal when noise is the only interference to be combatted are quite well understood at the present time. For brevity, the problem of determining only radial components of target position and velocity may be considered as a simple example.<sup>†</sup>

In principle, radial distance is determined by measuring the time which elapses between transmission of a radar signal and reception of its echo. The radial velocity component may be determined in two ways. One might measure the rate of change of the time delay which is already being used

---

<sup>†</sup> See references 2, 5, 35, and 39 for consideration of angular components of target motion.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

to provide range information. Or one might measure the frequency of the received echo and thereby deduce the Doppler frequency shift which the signal experienced upon reflection from the target.

It is known that the accuracy with which both target distance and velocity may be determined by these methods is primarily a function of the detailed shape of the transmitted radar waveform. Unless the received echo is overwhelmingly strong with respect to the noise, the accuracy of position and velocity measurement is only secondarily affected by received signal strength.

This discussion has, however, assumed an echo of sufficient energy to be detected. In fact, for white-noise interference of given power spectral density, the detectability of a radar echo does depend only upon its total energy, and not at all upon its particular waveform.† For a target of given reflectivity (or radar cross-section), therefore, detectability ultimately depends only upon the energy of the transmitted signal.

The significant point for the present discussion is that the detectability of a target and the accuracy with which its radial position and velocity may be measured depend upon different and non-conflicting attributes of the transmitted signal. Thus it is possible to conceive of designing a signal with the requisite complexity for achieving acceptable measurement accuracy, and then transmitting it with sufficient energy for the echo to be detected.

---

† This performance is true in theory and is rather closely approached in practice.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

For observing isolated targets in white-noise interference, therefore, system performance depends upon three things. They are: first, the transmitter power; second, the transmitted waveform; and finally, the manner in which the received echo is filtered or otherwise processed. In fact, for isolated echoes in white-noise interference, system performance tends to be limited only by the degree of system complexity which can be made to function with existing technology.

However, when clutter is admitted as one of the components of unwanted interference, then the situation changes completely.

"Clutter" is the term given, in radar, to the unwanted interfering echo which arises when the transmitted signal is reflected from (usually) extended objects in the radar field of view. The earliest examples of clutter arose by reflection of the signal from the nearby terrain, or ocean surface, upon which the radar was situated. Clutter returns are also received from the clouds of "chaff" (lightweight reflecting dipoles, or tinsel) which are dispersed by an offender to nullify a defender's radar effectiveness. Finally, one might consider an echo from the turbulent, ionized wake region behind a high-velocity object which is entering the earth's atmosphere, to be a form of clutter which might interfere with observation of the object itself.

In all cases the clutter return arises by reflection from a region which is extended throughout some region of space and which may have some sort of internal velocity structure. These detailed aspects of the clutter source will be considered in later sections.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

There is one attribute of the clutter return, however, which has far-reaching significance and which ultimately provides the motivation for the present research. That attribute is the readily apparent fact that, for a given source of clutter, the power of the clutter return is directly proportional to the transmitter power. Therefore both received signal power and received clutter power increase, and decrease, in exact step with any changes of transmitter power.

Consequently, when clutter interference predominates, it is impossible to cause the signal echo to stand out from the interference simply by transmitting a larger signal. This means that one adjustable parameter in system design, viz., transmitter power, has lost its effectiveness in controlling signal detectability. In fact one expects that, for any given signal waveform, the ratio of received echo power to received clutter power will be independent of transmitter power and will depend, instead, upon the ratio of the reflectivities of the target and the clutter source.

For any particular source of clutter interference, therefore, one can expect overall system performance to depend upon, and be influenced by, only two things, viz., (i) the shape of the transmitted waveform, and (ii) the manner in which the received waveform is filtered or processed.

This represents a considerable restriction from the preceding case of detection in white noise. Indeed, because of the restricted control over system performance, it is no longer clear whether one can simultaneously achieve good signal detectability together with specified range and velocity measurement accuracies. One might believe, with some justification from the existing literature which will soon be discussed, that clutter source reflectivity, for example, will



## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

set a limit to the signal detectability which can be achieved if specified measurement accuracies are also to be attained. The question however is essentially unresolved and provides one of the basic motivations for this research.

With the possibility of conflicting requirements being imposed upon system performance in clutter, it is also to be expected that selection of a transmitter waveform, and design of a signal processor for the receiver, will be influenced by the priority accorded various requirements.

In the present research, greatest interest centers about signal detection, with less interest attached to range and velocity measurement accuracy. The reverse order of these priorities has more often been considered. However, the order adopted for this research agrees with the fact that an echo must at least be detectable before its parameters can be measured with any confidence. In severe clutter environments, moreover, it is exactly the lack of signal detectability which provides the greatest practical problems.

The purpose of this research is therefore to investigate the properties of signal processors which yield optimum detection of radar echoes in noise and clutter interference.

One area of particular interest is the comparison between performance of processors which are optimum for noise and clutter interference, and processors which are optimum for noise alone but which are also experiencing clutter interference. The general question to be answered is whether the improved detection performance which can be achieved by processors suitably optimized for clutter and noise, is worth the additional processor complexity, or possible degradation of other aspects of performance, which can be expected. Ex-

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

amples which illustrate the conflicting considerations that are involved will be seen.

It is appropriate at this point, before considering the substance of the dissertation, to indicate briefly the organization of the material to be presented.

This chapter has given a qualitative description of the problem, together with the motivation for its study. Chapter two, to follow, provides a discussion of work already reported by other authors, on topics related in some manner to the present problem.

Against this background, Chapter III presents a summary of the major results and contributions which have arisen from this research. The remaining chapters provide the detailed substantiation of the results described in Chapter III.

Chapters IV and V together present a careful formulation of the analytical framework for this research. It is Chapter IV which defines the class of "optimum" signal processors being investigated in the research. In Chapter IV, also, will be found the basic integral equation which must be solved for the optimum processor weighting function. Chapter V gives expressions for the general ambiguity functions which are appropriate for describing optimum system performance.

Chapters VI through VIII comprise a group which takes up the problem of solving the basic integral equation. General forms of solution are presented in Chapter VI, while approximate solutions for large and small clutter are given in Chapter VII. For clutter from a strictly stationary source, Chapter VIII provides a method for replacing the basic integral equation by an equivalent difference-differential equation. This replacement may, or may not, represent a simplification of the problem.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Chapter IX stands by itself in presenting an upper bound to detection performance which is achievable by the optimum processor.

The final group of chapters, X through XIII, contains specific examples of optimum processors, and their performance, for a variety of specific cases which have different signal waveforms and clutter source models.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## II. HISTORICAL BACKGROUND

Three significant components of the problem considered in this research have already been mentioned, namely

- (i) a radially extended, possibly turbulent, reflecting medium which gives rise to clutter interference;
- (ii) the waveform which is transmitted; and
- (iii) the data processor, or filter, which characterizes the radar receiver.

One conclusion of this research is that, in addition, one must also consider

- (iv) the presence of noise.

These four factors have already been considered in various combinations, and for different purposes by numerous investigators. The present section is a brief discussion of this previous research. The purpose is to identify points of similarity and difference with the contents of this dissertation, in order that the research reported here may be placed into proper context. One conclusion which will also emerge is that relatively little of the research reported to date is directly antecedent to the research described in this dissertation.

The literature which has arisen concerning radar systems required to operate with inputs including clutter-type waveforms may be divided into five categories, according to the

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

emphasis given various parts of the problem.† There have been investigations primarily concerned with:

1. detection of time- and frequency-dispersed signals in noise;
2. performance of matched filter receivers in clutter;
3. design or discovery of "good" waveforms for use with matched filter receivers;
4. design of systems for moving-target-indication; and
5. performance of optimum receivers, in the sense used to describe the present research.

The remainder of this chapter is a consideration of each of these categories.

### A. DETECTION OF TIME- AND FREQUENCY-DISPERSED SIGNALS

Typical of research in this category are papers by Price,<sup>32</sup> Price and Green,<sup>33</sup> and Kailath,<sup>18</sup> where the problem is to detect a signal after it has been reflected from a spatial distribution of (assumed) randomly moving scatterers. Such signals have been assumed, for example, to arise because of tropospheric scattering of radio signals,<sup>32</sup> or reflection of radio signals from large areas of a rough planetary surface.<sup>33</sup>

The assumption of "independently moving scatterers" is the characteristic assumption for describing the source of a time- and frequency-dispersed echo. The same assumption is used for the clutter source for this dissertation.

---

† Attention is restricted to papers having to do primarily with problems of signal detection. Literature concerned with other problems, such as that by Krinitz<sup>23</sup> on system design for mapping the clutter source, is entirely beyond the purview of the present discussion.

## COLOMBIA UNIVERSITY--ELECTRONICS RESEARCH LABORATORIES

Note, however, that in this first category the basic goal is to detect the time- and frequency-dispersed echo, because it represents the signal component of the received waveform; the interference is assumed to be noise alone. One major concern of the three authors mentioned is the investigation of optimum processors for signal detection in the circumstances described.

The problem, however, is rather opposite to the problem considered in the present research. Here the aim has been to reject the time- and frequency-dispersed echo, as well as possible, because it represents an interference component of the total received waveform. This qualitative difference in viewpoint has as its direct consequence the fact that the two types of problem lead to integral equations for their solution which have essentially different structures. The solutions to the two problems are, therefore, not readily exchanged. Consequently, the results of research in this first category seem not to be directly applicable to the present problem.

### B. PERFORMANCE OF MATCHED FILTER RECEIVERS IN CLUTTER

A "matched filter" receiver is the optimum receiver for detecting signals which are received in white-noise interference.<sup>†</sup> When clutter, however, is present as part of the interfering waveform, the matched filter receiver is no longer optimum for signal detection. Its performance in a clutter environment is, nevertheless, well understood and has been described by Westerfield, Prager, and Stewart,<sup>46</sup> and Fowle, Kelly, and Sheehan,<sup>13</sup> for examples.

---

<sup>†</sup> See Turin<sup>43</sup> or North.<sup>30</sup>

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The matched filter approach to receiver design yields a receiver whose detection performance, and range and velocity measurement accuracy as well, can be related simply and directly to system and clutter parameters. The restriction to a matched filter, however, permits no indication of possible performance improvement which might be obtained by a receiver which is truly, or even partially, optimized against the clutter environment.

This second category of previously reported research is, therefore, of interest for the present research mainly as a source of reference performance data for comparison purposes. Data from Westerfield, et al.,<sup>46</sup> appears in Chapter X of this dissertation.

### C. DESIGN OR DISCOVERY OF "GOOD" WAVEFORMS

The attempt to design matched filter receivers to work well in a clutter environment has led to much research on the design of waveforms which will provide specified distance and/or velocity measurement accuracies, while simultaneously permitting detection performance which won't be too seriously degraded by clutter interference.

Rihaczek provides a summary of the interrelated "problems of measurement precision, target resolution, and waveform design."<sup>37</sup> Price and Hofstetter<sup>34</sup> more recently have contributed new information on the subject of achievable waveform properties.

All effort in this area, however, seems to be confined to waveforms for use with matched filter receivers. The present author is unaware of research into the problem of designing or discovering "good" waveforms for use in clutter environments with optimum receiver systems.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### D. DESIGN OF SYSTEMS FOR MOVING TARGET INDICATION (MTI)

In this category is research such as that reported by White and Ruvin,<sup>47</sup> having to do with designing radar systems which will indicate the presence only of moving targets. Stationary targets, and this includes clutter sources, will hopefully not appear.<sup>†</sup>

The basic design premise here is that the radar receiver will be designed to reject those portions of the received spectrum where a significant fraction of the clutter energy lies. Signal energy in this region will likewise be rejected. However, the signals which are caused by moving targets will be shifted in frequency and will have the bulk of their energy outside of the rejection region. This energy will be accepted by the receiver and the moving target detected.

Such receivers represent a midway point in design philosophy between the simple matched filter receiver described above and the optimum processor of the present research. The improvement in clutter rejection, over that achieved by the simple matched filter, is obtained in a relatively simple and natural manner. However, the ad hoc nature of the solution, together with the lack of a comprehensive and analytical problem formulation, prevents a general determination of, or insight into, just how well clutter might possibly be rejected and at what cost.

### E. PERFORMANCE OF OPTIMUM RECEIVERS IN CLUTTER

The single forerunner to the present research which is known to this author is the original paper by Urkowitz.<sup>45</sup> It is, likewise, the sole member of this fifth category.

---

<sup>†</sup> References 10 and 14 consider some practical restraints upon MTI performance.



## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Urkowitz found the optimum filter for detecting a signal, actually a rectangular pulse, in the presence of clutter from a uniformly extended spatial distribution of motionless reflecting particles, in the complete absence of noise. He found that under these circumstances the optimum filter for detecting the rectangular pulse is a waveform differentiator followed by a unity-gain recirculating-delay-line filter.

He then compared the performance of a receiver using band-limited approximations to each of these components to the performance of a receiver with a band-limited "flat" intermediate frequency filter. He found that the detection performance of the band-limited approximation to the optimum receiver increased linearly with receiver bandwidth. The performance of the band-limited, more conventional receiver, however, only increased until the signal bandwidth itself was approached. Above that point the performance remained constant as bandwidth was increased. These results, particularly the unboundedly good performance of the optimum processor, may now in retrospect be attributed entirely to the assumed lack of noise in Urkowitz' problem.

He concluded, from qualitative considerations, that although, in the presence of noise, the recirculating-delay-line portion of the processor "increases the signal-to-clutter ratio by another three decibels, ... it has a disastrous effect on the signal-to-noise ratio. The use of such a filter makes the problem of noise at least as severe as that of clutter."

Thus, Urkowitz' research<sup>45</sup> is the first particular solution for an optimum processor in a clutter environment. The problem formulation, however, is quite restricted, being

## COLUMBIA UNIVERSITY--ELECTRONICS RESEARCH LABORATORIES

almost entirely limited to the case already described. It does not have generality enough to include either

- (i) spatial variation of the mean radar cross-section of the clutter source;
- (ii) doppler dispersion introduced by localized motions within the clutter source;
- (iii) effects of noise in determining, and actually controlling, the optimum solution; or
- (iv) effects of target velocity upon signal echo detectability.

There is, in consequence, almost no consideration of optimum solutions for these more general circumstances†

The conclusion, therefore, is justified that previously reported research includes only scant mention of optimum processors for detecting radar echoes in a mixture of noise and clutter interference.

---

† The exception to this demurrer is Urkowitz' brief commentary on noise, part of which has already been quoted.

### III. RÉSUMÉ OF RESULTS AND CONCLUSIONS

This research has been motivated by the need to discover the extent to which radar system performance might be improved by employing an optimum receiver in the presence of noise-plus-clutter, instead of the simpler "matched filter" receiver,<sup>†</sup> and to discover the costs of such an approach in either increased system complexity, or degradation of other aspects of system performance.

The contributions of this research to the existing state of knowledge in this area occur in four major categories:

1. The problem formulation itself, which leads to an essentially new integral equation in the realm of signal detection theory.
2. General forms of solution of the integral equation.
3. A bound upon possible performance improvement.
4. A variety of solutions for particular cases which involve different transmitter waveforms and clutter sources.

These four categories correspond also to the order of presentation of the research in chapters IV through XIII of this dissertation. The remainder of this chapter is a presentation and discussion of the major results and conclusions which have arisen out of the present research. References to subsequent chapters will indicate the origins within the present research of the conclusions being discussed.

---

<sup>†</sup> the optimum receiver in the presence of noise alone.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### A. PROBLEM FORMULATION

In the present research, the signal echo to be detected is assumed to arise by reflection of the transmitted waveform from a moving, non-scintillating, essentially point-like target. The clutter component of interference, on the other hand, is assumed to arise by reflection from a spatial distribution of independent scattering centers, possibly in random motion. The formulation is sufficiently general to accommodate

- i) a transmitted waveform with arbitrary amplitude and phase modulations;
- ii) an arbitrary level of white-noise interference;
- iii) a clutter source which may have different mean reflectivities at different locations, and which may yield an echo with Doppler frequency dispersion because of local source motions.

A formulation of this generality is a necessity because all the factors mentioned can and do affect system performance, sometimes quite markedly.

The combined noise and clutter interference which arises from this formulation is a Gaussian process. One appropriate measure of signal detectability is consequently the signal-to-interference ratio, irrespective of the source of the interference. The "optimum" processors of this research act to maximize the signal-to-interference ratio (and, hence, the signal detectability) at their output. The integral equation which determines the optimum processor is derived and presented in chapter four.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The problem formulation, in the generality described, and the integral equation which arises from it are essentially new to the literature in signal detection theory.

The integral equation which determines the optimum processor weighting function  $w(t; \rho_0, f_0)$  is given by†

$$N_0 \cdot w(t_1; \rho_0, f_0) + \int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 = m(t_1; \rho, f_0) \quad (3.1)$$

$$\text{where } \mathcal{K}_c(t_1, t_2) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}(\rho, f) m(t_1; \rho, f) m^*(t_2; \rho, f) d\rho df \quad (3.2)$$

$m(t; \rho, f)$  = modulation function for an echo received after a range delay of  $\rho$  seconds, with a Doppler frequency shift of  $f$  cycles per second.

$\mathcal{E}(\rho, f)$  = energy dispersion function for the clutter source

$N_0$  = white-noise power spectral density

The essential novelty of this equation resides, first, in the particular structure of the kernel  $\mathcal{K}_c(t_1, t_2)$  and, second, in the fact that the modulation function  $m(t; \rho, f)$  appears not only on the right-hand side in the conventional role of a "forcing function", but also within the defining equation for the kernel  $\mathcal{K}_c(t_1, t_2)$ .

In consequence of the second fact, one cannot expect that the "forcing" function  $m(t; \rho_0, f_0)$  and the integral equation solution, or "response" function,  $w(t; \rho_0, f_0)$  will be linearly related. Because the methods of linear system

† Chapter IV

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

analysis are, therefore, not applicable in relating these functions, optimum system performance is not easily related to attributes of the transmitter modulating function.

Direct inspection of the preceding integral equation also reveals that the two terms on the left which involve the unknown function  $w(t; \rho_0, f_0)$  are proportional to the noise level,  $N_0$ , and the clutter level,  $\mathcal{E}(\rho, f)$ , respectively. One therefore expects that the form of the optimum solution  $w(t; \rho_0, f_0)$  will depend upon the ratio of the clutter and noise levels. This conclusion will be amplified later in this chapter. For the present it suffices to observe that, in general, one can expect the means chosen to combat clutter in any particular circumstance to depend, not only upon the detailed nature of the clutter, but also upon the (always present) noise level which is accompanying the clutter.

The most elementary conclusion has, thus far, not been explicitly mentioned. It is simply that the processor which is optimal for detection in noise alone is sub-optimal when clutter interference is added to the noise. Worded differently, the optimum processor for detection in clutter-plus-noise interference is not, in general, the "matched" filter whose impulse response duplicates (with time reversal) the signal to be detected.

### B. INTEGRAL EQUATION SOLUTIONS

The solution to the preceding integral equation can be written in many different forms. In this dissertation three forms of solution are presented† which are valid for all reasonable and "non-pathological"  $N_0$ ,  $\mathcal{E}(\rho, f)$ , and  $m(t; \rho, f)$ .

---

† Chapter VI

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Two other forms of solution are presented† for situations where noise is either the dominant interference component, or is neglected entirely. Finally, a method is presented†† whereby, if the modulation function  $m(t; \rho, f)$  has certain properties, and if the clutter source is motionless, one can convert the preceding integral equation to an equivalent difference-differential equation with, generally, variable coefficients.

The three general forms of solution which are presented††† arise by application of the well known theory of linear integral equations to the present case. They may be characterized, briefly, as:

1. A solution, according to Hilbert-Schmidt theory, in terms of eigenfunctions and eigenvalues of the kernel  $\mathcal{K}_c(t_1, t_2)$ .
2. A solution, according to Fredholm's theory, which depends upon iterated kernels and which, at any time instant, is a rational function of the clutter-to-noise ratio.
3. A solution, possible only because of the particular structure of the kernel  $\mathcal{K}_c(t_1, t_2)$ , which gives  $w(t; \rho_0, f_0)$  as a superposition of delayed and Doppler-shifted echoes  $m(t; \rho_i, f_j)$ .

The practical application of any of these solutions to a particular case, however, still requires the solution of formidable problems.

In the eigenfunction solution the first problem is, of course, to discover eigenfunctions and eigenvalues for the

---

† Chapter VII

†† Chapter VIII

††† Chapter VI

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

kernel  $\mathcal{K}_C(t_1, t_2)$ . Even assuming this to be possible one cannot, in view of equation (3.2) for  $\mathcal{K}_C(t_1, t_2)$ , expect that either the eigenfunctions or eigenvalues will bear any simple relation to either the signal, represented by  $m(t; \rho, f)$ , or to the clutter source, represented by  $\mathcal{E}(\rho, f)$ .

The following bounds for the eigenvalues  $\lambda_j$  have, however, been derived

$$0 \leq \lambda_j \leq 2 \cdot \max_{(\rho, f)} \mathcal{E}(\rho, f). \quad (3.3)$$

These bounds make possible the direct evaluation of the performance improvement bound which will soon be considered by itself.

The Fredholm solution, which was second in the preceding list of the three general solutions, is at least given directly in terms of operations upon the basic kernel  $\mathcal{K}_C(t_1, t_2)$ . If the functions  $m(t; \rho, f)$  and  $\mathcal{E}(\rho, f)$  are such that, through direct integration, the iterated kernels which depend upon  $\mathcal{K}_C(t_1, t_2)$  can be found, then the means for a solution are at hand. This mode of solution was not attempted for any particular case in this research, simply because of the difficulties of iterated integration of functions which, for practical cases, are not necessarily simple.

This solution, however, is one which exhibits explicit dependence upon the clutter-to-noise-ratio parameter  $\mathcal{R}_C$  defined in this research by

$$\mathcal{R}_C = \frac{2 \hat{\mathcal{E}}_C}{N_0}, \quad (3.4)$$



## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

where 
$$\hat{\mathcal{E}}_c = \max_{(\rho, f)} \mathcal{E}(\rho, f) \quad (3.5)$$

The parameter  $\rho_c$  appears in a natural fashion in other places in this research, also, but most notably in the performance improvement bound. It seems to be a noteworthy parameter for describing mixed clutter and noise interference.

The third general form of solution presented in this dissertation gives  $w(t; \rho_o, f_o)$  as the weighted (double) sum of delayed and doppler shifted echos  $m(t; \rho_i, f_j)$ . The form is appealing but determination of the appropriate weights for the individual summands rests upon solution of an auxiliary (double) integral equation. Only in the case where noise is neglected did this third form of solution, therefore, yield results.

When noise is totally neglected, the optimum processor acquires very simple characteristics.† In the suggestive terminology of vector analysis, the optimum weight function may be described as being "orthogonal" to essentially all clutter components, while having non-zero, or unit, "projection" upon the desired echo. The weight function may also be described as being "reciprocal" to the desired echo.

It is pointed out in chapter seven, however, that the zero-noise optimum processor achieves these properties only by having a certain "super-resolution" capability with respect to the desired echo. The zero-noise optimum processor is therefore characterized by bandwidths much larger than the signal bandwidth and, at least in one case, by durations which include the total clutter return.

---

† Chapter VII B.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Because these properties can profitably be exploited only in the complete absence of noise, one concludes that noise is never a negligible factor in considering systems for clutter suppression. This conclusion is strengthened by the various results for special cases which invariably show that the presence of even "small" noise levels (when compared to "large" clutter levels) greatly modify the form of the zero-noise optimum processor and reduce its performance.

One further concludes that the optimum weight function for small noise levels is not, in general, merely a small perturbation of the optimum processor for zero noise. Unfortunately, while the case of large clutter and small noise is the most interesting one from a practical point of view, it is the case of zero noise which is most often easier to solve. The inability to approximate one solution by the other, because one is not a small perturbation of the other, is a far-reaching technical fact which directly adds to the difficulties of solving the present problem.

The converse case of detection in interference when noise is dominant and clutter interference is relatively small provides a much simpler solution.<sup>†</sup>

The important fact, however, which emerges from comparison of the small-noise and large-noise solutions, is the great dependence of the form of the optimum processor upon the relative levels of clutter and noise interference.

The final approach<sup>††</sup> to solving the basic integral equation (3.1) which was tried in this research was based upon a method due to Miller and Zadeh<sup>29</sup> for solving integral equa-

---

<sup>†</sup> Chapter VII A.

<sup>††</sup> Chapter VIII

tions with kernels somewhat similar to  $K_c(t_1, t_2)$  given by equation (3.2). With appropriate modifications, to accommodate the somewhat more general radar waveforms which appear in the present context, their method was adjusted to be applicable to equations (3.1) and (3.2). The end result is a procedure for deriving an equivalent difference-differential equation for the unknown weight function  $w(t; \rho_0, f_0)$ . Although it is difficult to characterize this transformation of the problem as a "simplification," the method was applied in one of the particular cases considered.

### C. PERFORMANCE IMPROVEMENT BOUND

It will be recalled that one major question motivating this research was the extent to which optimum system performance might exceed the performance of a matched filter receiver in a clutter-plus-noise interference environment. One answer to this question is provided by the following inequality†

$$\left(\frac{S}{I}\right)_{\text{opt}} \leq \left(\frac{S}{I}\right)_{\text{mf}} \cdot B(\mathcal{R}_c) \quad (3.6)$$

where the scalar factor  $B(\mathcal{R}_c)$  is given by

$$B(\mathcal{R}_c) = \frac{1}{4} \left[ (1+\mathcal{R}_c)^{\frac{1}{2}} + (1+\mathcal{R}_c)^{-\frac{1}{2}} \right]^2 \quad (3.7)$$

and  $\mathcal{R}_c$  is the clutter-to-noise-ratio parameter given earlier in equations (3.4) and (3.5).

$B(\mathcal{R}_c)$  is therefore an upper bound to the improvement in signal-to-interference ratio which may be achieved by departing from a matched processor and choosing a processor suit-

---

† Chapter IX

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

ably optimized against the presence of clutter. Equations (3.6) and (3.7) are derived in chapter IX and have the full generality implied by their appearance. That is, the inequality (3.6) is valid irrespective of the signal modulation function  $m(t; \rho_0, f_0)$  or the detailed form of the clutter dispersion function  $\mathcal{C}(\rho, f)$ . The performance improvement bound  $B(\mathcal{R}_c)$  depends exactly and only upon the parameter  $\mathcal{R}_c$  and is valid over the entire scope of this research.

It is interesting to note that as  $\mathcal{R}_c$  approaches infinity, say for example because the noise level  $N_0$  is approaching zero, then

$$B(\mathcal{R}_c) \rightarrow \frac{1}{4} \cdot \mathcal{R}_c. \quad (3.8)$$

In effect this bears out previous remarks that the presence of noise is an inescapable consideration in clutter rejection systems. Were  $N_0$  actually zero, then  $B(\mathcal{R}_c)$  would be infinite and perfect clutter rejection might be achievable. In fact  $N_0$  is never zero, with the result that  $B(\mathcal{R}_c)$  is always finite and the previously mentioned "reciprocal" waveforms are seldom appropriate weighting functions for practical processors.

As  $\mathcal{R}_c$  approaches zero, on the other hand, one finds that

$$B(\mathcal{R}_c) \rightarrow 1 + \frac{1}{4} \cdot \mathcal{R}_c^2 \quad (3.9)$$

which indicates that for small clutter levels there is little advantage to departing from the matched filter receiver. Any advantage can only be proportional to the square of the already small  $\mathcal{R}_c$ .

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Conditions for which equality in (3.6) is actually achieved are given in the text,<sup>†</sup> but can be expected to be only rarely satisfied in practice.

### D. PARTICULAR CASES

Optimum processors have been derived for a variety of particular cases involving different modulation functions  $m(t; \rho_0, f_0)$  and dispersion functions  $\mathcal{E}(\rho, f)$ . The cases considered are:

1. Gaussian pulse echo in uniformly extended Gaussian clutter plus noise,
2. Gaussian pulse echo in clutter with Gaussian delay and Doppler profiles.
3. Rectangular pulse echo in uniformly extended clutter from a stationary source, with various noise levels.
4. Rectangular pulse echo in clutter from a stationary source of finite extent.

The solutions for these cases illustrate well the general remarks which have already been made. In the remainder of this chapter it will suffice to note various aspects of these particular solutions which might not otherwise be deduced from general considerations.

In the first case,<sup>††</sup> of a Gaussian pulse in uniformly extended Gaussian clutter, the basic integral equation is readily solved by means of Fourier transforms. In the frequency domain one can see that, in the absence of noise, the optimum processor emphasizes only those portions of the spec-

---

<sup>†</sup> Chapter IX.

<sup>††</sup> Chapter X.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

trum where the signal energy is relatively large compared to the clutter energy, somewhat in the manner of a system for moving-target-indication. One can also see the great changes in the solution which occur as the result of even small noise levels. Empirical performance improvements calculated from numerical data for this case indicate clearly the extent (about 30 db) to which the performance improvement bound  $B(\mathcal{R}_c)$  can exceed actual performance differences.

The second case considered† includes clutter interference with a mean intensity which varies with time because of the spatial variation of mean reflectivity of the clutter source. Because of the particular waveforms chosen, an analytical solution is possible. The optimum processor is found to be a time-variable filter, composed of a time-varying zero-memory amplifier followed by a stationary linear filter. That a time-variable processor is optimum for detecting echoes in statistically non-stationary interference is to be expected. It is not to be expected that the processor structure will be so simply defined.

The third case is solved†† starting from the equivalent difference-differential equation. The resulting zero-noise processor is a train of impulse doublets which provide an example of a reciprocal waveform for the rectangular pulse echo. In effect, detection is accomplished by sensing only the leading and trailing edges of the echo. Numerical results are given, showing the radically different processors which arise when small and then large noise levels are introduced.

---

† Chapter XI

†† Chapter XII

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

When clutter from a stationary source of finite extent is considered, the fourth and final case,<sup>†</sup> it is shown that the known properties of the reciprocal waveform from the preceding simpler case may be used to advantage in providing a solution to the final case.

The several particular solutions together illustrate one other general aspect of the basic integral equation. It has already been noted that the kernel  $\mathcal{K}_c(t_1, t_2)$  depends upon whatever functions are chosen for the modulation function  $m(t; \rho, f)$  and the dispersion function  $\mathcal{E}(\rho, f)$ . This leads to the situation, well illustrated in this research, that the particular means used to solve the integral equation will depend largely upon the particular case being considered.

---

<sup>†</sup> Chapter XIII

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### IV. PROBLEM FORMULATION

#### A. DESCRIPTION OF A RADAR SYSTEM

The major components of the system to be studied are a transmitter, a receiver, and a linear data processor. They are shown in the functional block diagram of Fig. 1 together with designations for the essential system waveforms.

##### 1. Transmitter

The single characteristic of the transmitter which is of interest in this study is the transmitted waveform represented by  $S(t)$ . This complex function of time has the form

$$S(t) = m(t)\exp\{j2\pi f_c t\} \quad (4.1)$$

where  $f_c$  is the unmodulated transmitter carrier frequency in cycles per second. The complex modulation function  $m(t)$  has the form

$$m(t) = a(t)\exp\{j\phi(t)\} \quad (4.2)$$

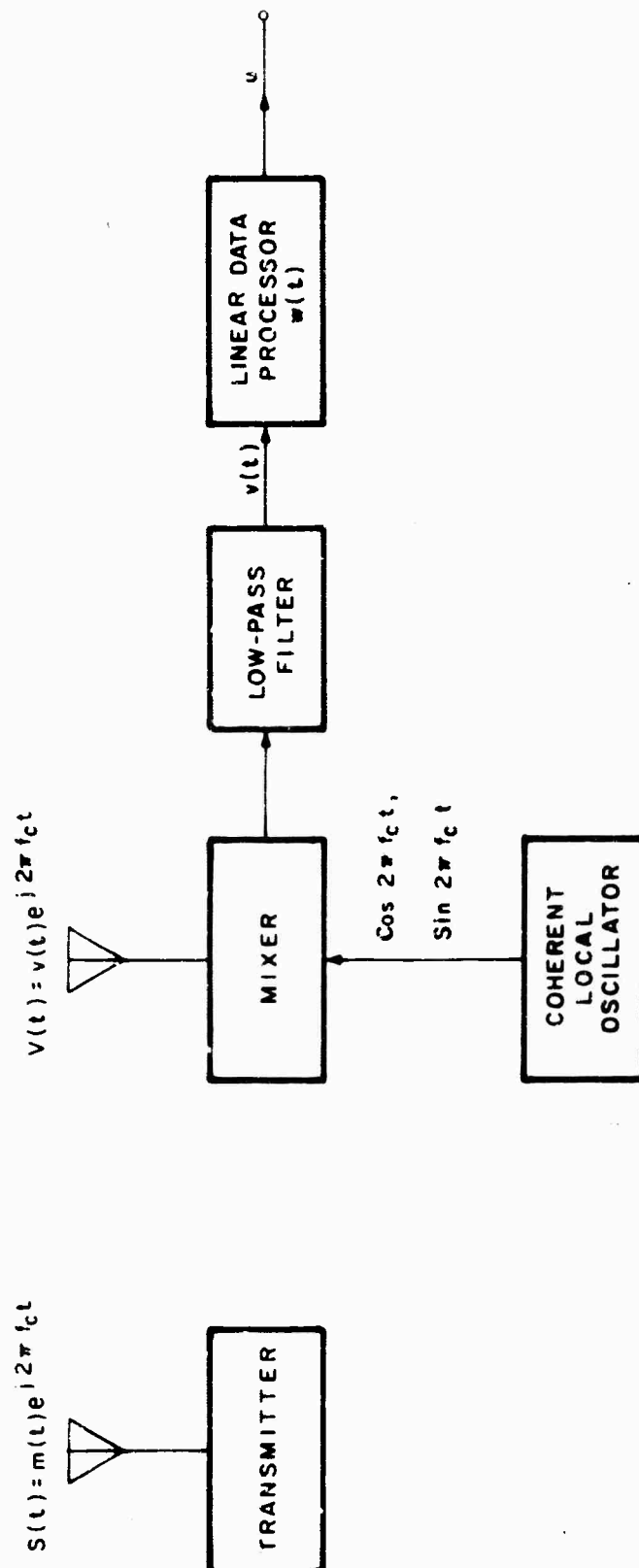
where  $a(t)$  = a real, carrier amplitude modulating function, and  $\phi(t)$  = a real, carrier phase modulating function.

This waveform representation is sufficiently general for the purposes of this study and permits consideration of transmitted signals having simultaneous phase and amplitude modulation.

The real waveform which would actually be transmitted is taken to be the real part of the complex  $S(t)$ . In the present instance the real component of  $S(t)$  is simply



COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES



A-194-S-0068

FIG. 1 A CANONICAL RADAR SYSTEM

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$\operatorname{Re}\{S(t)\} = a(t) \cos \{2\pi f_c t + \phi(t)\} . \quad (4.3)$$

The waveform at the receiver input, which is the sum of signal and interference, likewise may be represented as a complex envelope function modulating a carrier. If this carrier is taken to be identical to the transmitter carrier then the received waveform representation is

$$V(t) = v(t) \exp \{j2\pi f_c t\} . \quad (4.4)$$

### 2. Receiver

The primary function of the receiver for the purposes of this study is the extraction of the complex envelope function  $v(t)$ . Mathematically this is accomplished by supposing the existence of a signal  $\exp \{-j2\pi f_c t\}$ , in order that a simple product may yield

$$v(t) = V(t) \cdot \exp \{-j2\pi f_c t\} . \quad (4.5)$$

In actuality, this demodulation process may be accomplished by heterodyning the received waveform separately against  $\cos \{2\pi f_c t\}$  and  $\sin \{2\pi f_c t\}$  and then filtering out double-frequency components. The remaining signals will be quadrature components of  $v(t)$ .

### 3. Data Processor

It will be assumed, however it may be accomplished, that the receiver output is the complex function of time  $v(t)$ . It is in the data processor that whatever information may be present about radar targets is extracted from  $v(t)$  and made available for use as a system output.

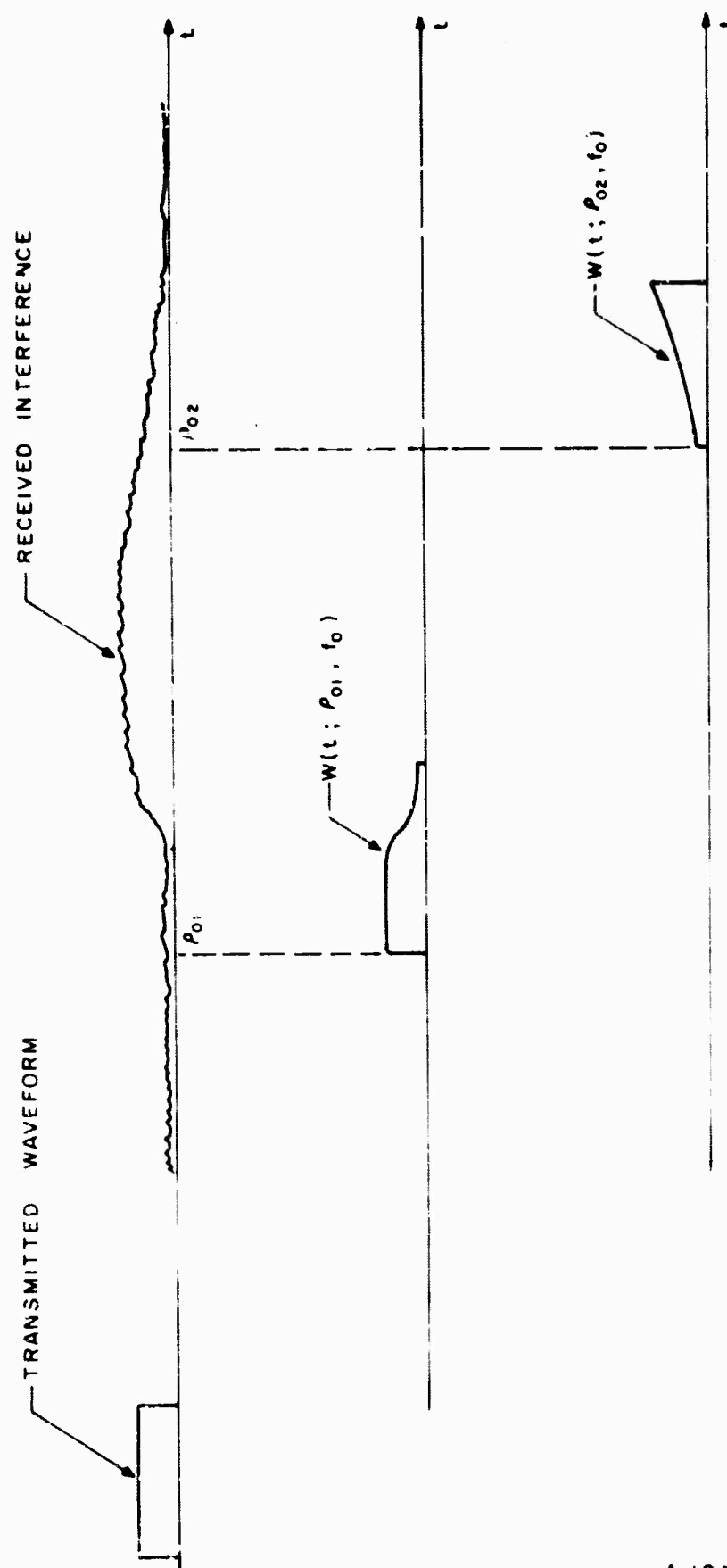
## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

It will further be assumed that the processor is designed to detect a radar echo which is received  $\rho_0$  seconds after transmission of  $S(t)$  and with a possible doppler shift of  $f_0$  cycles per second. The processor will be characterized by a weight function which is designated  $w(t; \rho_0, f_0)$ . The scalar processor output  $u$  is assumed to be a linear functional of possibly the entire record  $v(t)$  available to the processor input. The processor is therefore defined by the equation

$$u = \int_{-\infty}^{\infty} w^*(t; \rho_0, f_0) v(t) dt \quad (4.6)$$

This is a sufficiently general definition of a linear data processor for the present analysis, wherein attention can be confined to well-behaved functions. The completely general representation of a bounded linear functional as a Stieltjes integral which has appeared in the literature,<sup>4a</sup> and which arises out of Riesz' representation theorem,<sup>3b</sup> is not necessary here.

The notation for the processor weight function includes rather prominently the parameter set  $(\rho_0, f_0)$  for the echo to be detected. This is to show that the form of the weight function may well depend upon  $(\rho_0, f_0)$  in a non-trivial fashion when clutter interference is present. This is suggested in Fig. 2, where received clutter interference is shown following transmission of a hypothetical  $S(t)$ . The usual situation where the mean clutter energy is a function of time delay after transmission is also suggested. The essential point is that the clutter interference can have a distinctly non-stationary character as a function of time delay. The result is that the desired weight function  $w(t; \rho_{02}, f_0)$  for an echo with range delay  $\rho_{02}$  will not



A-194-S-0069

FIG. 2 CONCEIVABLE WEIGHT FUNCTIONS FOR NON-STATIONARY INTERFERENCE

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

necessarily be simply a time translation of  $w(t; \rho_{o1}, f_o)$  for an echo with delay  $\rho_{o1}$ . The echo parameters  $(\rho_o, f_o)$  are therefore a necessary part of the characterization of the data processor.

### B. THE NATURE OF CLUTTER

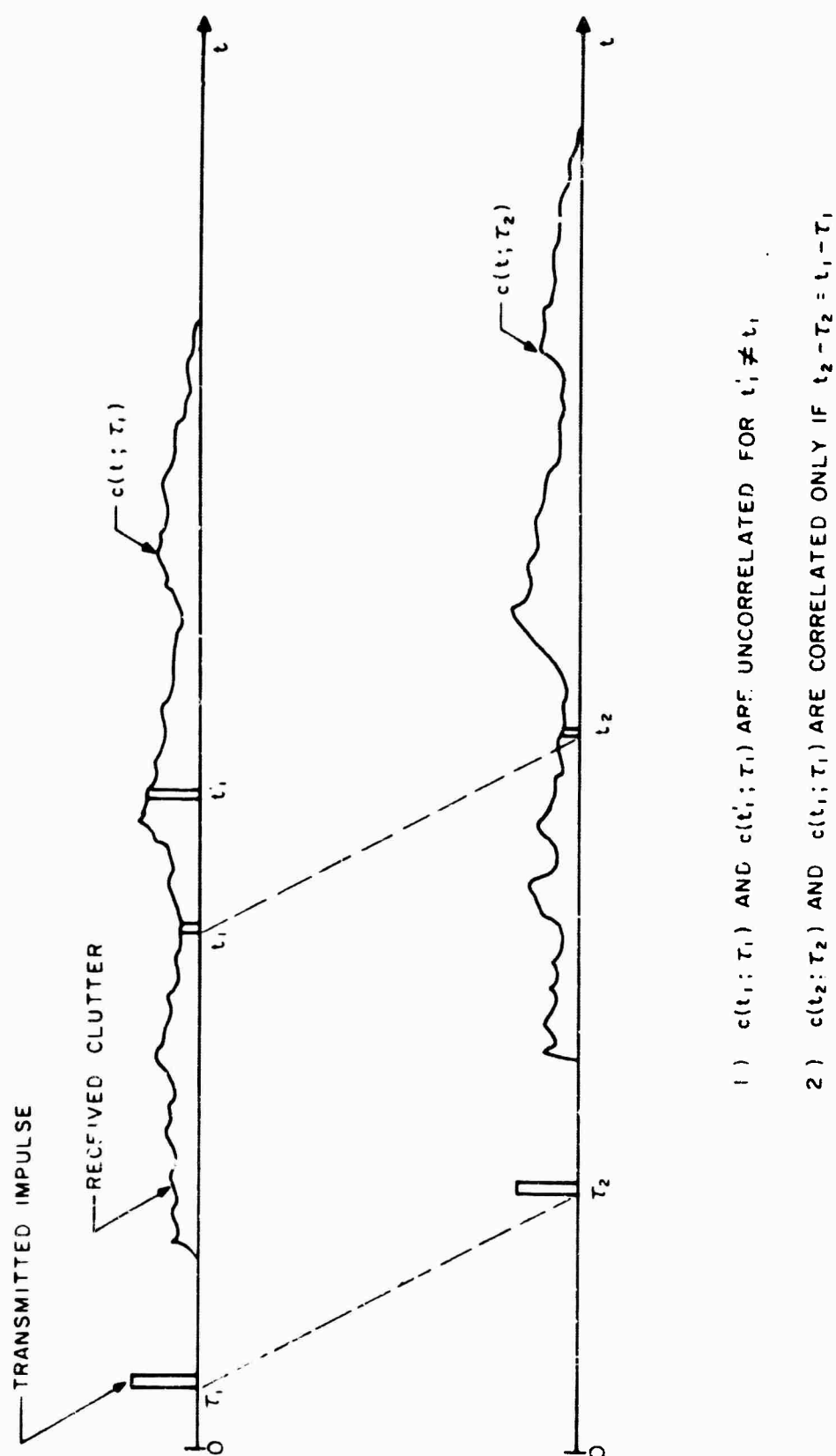
Clutter interference (or "reverberation" in SONAR) commonly arises by reflection of the transmitted waveform from a spatial distribution of reflecting, or scattering, points or regions. These scattering centers almost always have some relative motion with respect to one another, as well as with respect to the observer. The result is that the individual echoes comprising the clutter return also have some distribution of Doppler frequency shifts. Because the locations of individual scattering centers are changing, the clutter returns corresponding to successive transmissions will in general be different. The result is that received clutter waveforms are most conveniently characterized statistically.

In this section a common model for clutter will be described, the model which underlies this research.

Figure 3 shows two successive narrow transmitted pulses and the clutter waveforms resulting from each pulse. The transmissions occur at  $\tau_1$  and  $\tau_2$  seconds while the complex envelopes of the corresponding clutter returns are designated  $c(t; \tau_1)$  and  $c(t; \tau_2)$ . These returns are the superposition of numerous delayed and frequency shifted replicas of the transmitted pulses.

The first major feature of the model is the assumption that clutter components due to reflection of the same transmitted pulse from spatially separated points are uncorrelated. In terms of Fig. 3 this means, for example, that  $c(t_1; \tau_1)$

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES



A-194-S-0070

FIG. 3 CORRELATION IN CLUTTER RETURNS

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

and  $c(t'_1; \tau_1)$  are uncorrelated, because the amplitudes received at the different times  $t_1$  and  $t'_1$  correspond to reflections from different slant ranges.

The second major feature has to do with the reflection of different transmitted pulses from the same region of space at different times.<sup>†</sup> Here correlation between such returns is to be expected, unless the physical structure of the reflecting region has completely changed between interrogations. In Fig. 3, for example, correlation is expected between the clutter amplitudes  $c(t_1; \tau_1)$  and  $c(t_2; \tau_2)$  whenever the respective time delays  $(t_1 - \tau_1)$  and  $(t_2 - \tau_2)$  are equal.

The rate at which the structure of a collection of scattering centers changes is determined by the relative velocities of the individual points. The second assumption for the clutter model will be that reflections from scatterers having different velocities are uncorrelated.<sup>††</sup>

The major features of the assumed clutter interference have been discussed here in terms of the returns from successive interrogations by brief pulses. The later analysis, however, requires an expression for the correlation between received clutter amplitudes at any pair of times for an arbitrary illumination. That result is given in the following section. At the same time a precise description is given of the statistical structure of the clutter source.

---

<sup>†</sup> A large-scale translatory motion of the physical clutter source through space is here ignored as it can be compensated for in the receiver.

<sup>††</sup> When averaged over the ensemble of possible scatterer locations.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### 1. The Clutter Covariance Function

Let the arbitrary transmitted waveform be denoted by  $S(t)$ . It will be assumed that the echo of this waveform which is received after reflection from points with time delays between  $\rho$  and  $\rho + d\rho$ , and which induce Doppler frequency shifts between  $f$  and  $f + df$  may be represented by the term

$$a(\rho, f) \cdot S(t-\rho) \exp \{ j2\pi f(t-\rho) \} . \quad (4.7)$$

Here the change in amplitude and phase which  $S(t)$  experiences upon reflection is incorporated in the complex random coefficient  $a(\rho, f)$ .

The total clutter return  $C(t)$  may then be written in the form

$$C(t) = \sum_k \sum_{\ell} a(\rho_k, f_{\ell}) S(t-\rho_k) \exp \{ j2\pi f_{\ell}(t-\rho_k) \} \quad (4.8)$$

representing the sum of echoes from all ranges with all frequency shifts. If increments  $d\rho_k$  and  $df_{\ell}$  are defined to satisfy

$$\rho_k + d\rho_k = \rho_{k+1} \quad (4.9a)$$

and

$$f_{\ell} + df_{\ell} = f_{\ell+1} \quad (4.9b)$$

then Eq. (4.8) presents  $C(t)$  as the sum of contributions from a set of disjoint cells covering the  $(\rho, f)$  plane. For brevity the cell determined by the intervals  $(\rho_k, \rho_k + d\rho_k)$  and  $(f_{\ell}, f_{\ell} + df_{\ell})$  will be designated  $[d\rho_k, df_{\ell}]$ .

The covariance function of interest is defined by

$$\mathcal{K}_c(t_1, t_2) = \langle c(t_1) c^*(t_2) \rangle \quad (4.10)$$



COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

where  $c(t)$ , the complex envelope of  $C(t)$ , is given by

$$c(t) = C(t) \cdot \exp \{-j2\pi f_c t\} \quad (4.11)$$

and where the angle brackets indicate the expectation with respect to the ensemble of admissible clutter sources. When Eqs. (4.1), (4.8), and (4.11) are used with (4.10), the result is

$$\begin{aligned} \mathcal{K}_c(t_1, t_2) = & \sum_k \sum_\ell \sum_r \sum_s \left\langle a(\rho_k, f_\ell) a^*(\rho_r, f_s) \right\rangle \cdot m(t_1 - \rho_k) m^*(t_2 - \rho_r) \\ & \cdot \exp \left\{ j2\pi f_\ell (t_1 - \rho_k) + j2\pi f_c (t_1 - \rho_k) - j2\pi f_c t_1 \right. \\ & \left. - j2\pi f_s (t_2 - \rho_r) - j2\pi f_c (t_2 - \rho_r) + j2\pi f_c t_2 \right\}. \end{aligned} \quad (4.12)$$

It is assumed for the source that if the cells  $[d\rho_k, df_\ell]$  and  $[d\rho_r, df_s]$  are disjoint, then

$$\left\langle a(\rho_k, f_\ell) a^*(\rho_r, f_s) \right\rangle = 0. \quad (4.13)$$

On the other hand, if the cells  $[d\rho_k, df_\ell]$  and  $[d\rho_k, df_s]$  coincide it is assumed that the ensemble expectation may be written

$$\left\langle a(\rho_k, f_\ell) a^*(\rho_k, f_\ell) \right\rangle = 2 \mathcal{E}(\rho_k, f_\ell) d\rho_k df_\ell \quad \dagger \quad (4.14)$$

where  $\mathcal{E}(\rho_k, f_\ell)$  represents the average energy returned from the cell  $[d\rho_k, df_\ell]$  at  $(\rho_k, f_\ell)$  for incident signals of unit energy.

<sup>†</sup> Extended reflecting regions with these assumed characteristics have been considered in References 33 and 46.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

It is worth noting that the reflected energy is proportional to the cell dimensions, an assumption consistent with the notion of returns from adjacent points being uncorrelated. As a cell dimension is increased one would expect the additional signal components to add incoherently to the previous components. In this situation one expects signal energies, rather than amplitudes, to add linearly.

When Eqs. (4.13) and (4.14) are introduced into (4.12) only a double sum remains for  $K_c(t_1, t_2)$ , namely

$$K_c(t_1, t_2) = \sum_k \sum_\ell \mathcal{E}(\rho_k, f_\ell) \cdot m(t_1 - \rho_k) m^*(t_2 - \rho_k) \cdot 2 \cdot \exp \{ j2\pi f_\ell (t_1 - t_2) \} \cdot d\rho_k df_\ell. \quad (4.15)$$

In the limit, as  $d\rho_k$  and  $df_\ell$  approach zero, the double sum is assumed to approach a limit which defines the corresponding Riemann integral. Therefore, in the limit

$$K_c(t_1, t_2) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(t_1 - \rho) m^*(t_2 - \rho) \mathcal{E}(\rho, f) \exp \{ j2\pi f (t_1 - t_2) \} d\rho df. \quad (4.16)$$

This is the result which will be used. It describes the relevant statistical properties of the clutter interference at the input to the data processor.

To accord with notation previously introduced for the processor weight function, and to simplify expressions which will appear, it is convenient to define  $m(t; \rho, f)$  by

$$m(t; \rho, f) = m(t - \rho) \exp \{ j2\pi f (t - \rho) \}. \quad (4.17)$$

It may be verified that the preceding equation for  $K_c(t_1, t_2)$  is equivalent to

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$K_c(t_1, t_2) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}(\rho, f) m(t_1; \rho, f) m^*(t_2; \rho, f) d\rho df . \quad (4.18)$$

### C. THE OUTPUT SIGNAL-TO-INTERFERENCE RATIO

The signal-to-interference ratio at the data processor output is the quantity of major interest in this study.

It will be supposed that the ultimate task of the system is to detect point targets of unvarying reflectivity (i.e., not "scintillating") with unknown range and velocity. The interference is the sum of clutter previously described and noise with uniform power spectral density over the receiver bandwidth. The signal-to-interference ratio at the data processor output, designated  $\frac{S}{I}$ , will be derived for these circumstances.

Let the complex representation of the waveform at the receiver input be designated  $V(t)$  and have the form

$$V(t) = R(t; \rho_o, f_o) + I(t) \quad (4.19)$$

where  $R(t; \rho_o, f_o)$  = deterministic signal component  
from point reflector at range  
delay  $\rho_o$  with doppler shift  $f_o$ .

$I(t)$  = stochastic interfering waveform.

The received signal component is assumed to be the usual delayed and Doppler shifted replica of the transmitted waveform, that is

$$R(t; \rho_o, f_o) = A \cdot S(t - \rho_o) \cdot \exp \{ j 2 \pi f_o (t - \rho_o) \} . \quad (4.20)$$

It is assumed that  $m(t)$  is normalized to yield

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$\int_{-\infty}^{\infty} m^*(t)m(t)dt = 1 \quad (4.21)$$

so that for the received echo  $R(t; \rho_o, f_o)$

$$\int_{-\infty}^{\infty} R^*(t; \rho_o, f_o)R(t; \rho_o, f_o)dt = A^2 \quad (4.22)$$

Thus it follows that

$$A^2 = 2\mathcal{E}_s \quad (4.23)$$

where  $\mathcal{E}_s$  is the total energy of the received signal echo.

The data processor acts upon the complex envelope of  $R(t; \rho_o, f_o)$  to yield a scalar, designated  $r(\rho_o, f_o)$ . The signal component of the processor output is therefore given by (see Eq. 4.6)

$$r(\rho_o, f_o) = \int_{-\infty}^{\infty} w^*(t; \rho_o, f_o)R(t; \rho_o, f_o)\exp\{-j2\pi f_c t\}dt \quad (4.24)$$

or, when Eqs. (4.1) and (4.17) are employed,

$$r(\rho_o, f_o) = A \int_{-\infty}^{\infty} w^*(t; \rho_o, f_o)m(t; \rho_o, f_o)dt \cdot \exp\{-j2\pi f_c \rho_o\} \quad (4.25)$$

The intensity of this component is given by

$$|r(\rho_o, f_o)|^2 = A^2 \left| \int_{-\infty}^{\infty} w^*(t; \rho_o, f_o)m(t; \rho_o, f_o)dt \right|^2 \quad (4.26)$$

The received interference component has complex envelope  $i(t)$ , derived in the established manner:

$$i(t) = I(t) \cdot \exp\{-j2\pi f_c t\} \quad (4.27)$$

For interference of arbitrary nature the interference component in the processor output will be designated  $i(\rho_o, f_o)$

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

and written

$$i(\rho_o, f_o) = \int_{-\infty}^{\infty} w^*(t; \rho_o, f_o) i(t) dt \quad (4.28)$$

Its mean intensity is given by

$$\begin{aligned} \langle i(\rho_o, f_o) i^*(\rho_o, f_o) \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1; \rho_o, f_o) w(t_2; \rho_o, f_o) \\ &\cdot \langle i(t_1) i^*(t_2) \rangle dt_1 dt_2 \end{aligned} \quad (4.29)$$

where the expectation on the right has been brought inside the integral. If one defines the covariance function  $\mathcal{K}(t_1, t_2)$  by

$$\mathcal{K}(t_1, t_2) = \langle i(t_1) i^*(t_2) \rangle \quad (4.30)$$

then the following formula is obtained:

$$\begin{aligned} \langle i(\rho_o, f_o) i^*(\rho_o, f_o) \rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1; \rho_o, f_o) w(t_2; \rho_o, f_o) \\ &\mathcal{K}(t_1, t_2) dt_1 dt_2 \end{aligned} \quad (4.31)$$

The signal-to-interference ratio is the ratio of output signal intensity to mean output interference intensity,

$$\frac{S}{I} = \frac{|r(\rho_o, f_o)|^2}{\langle i(\rho_o, f_o) i^*(\rho_o, f_o) \rangle}, \quad (4.32)$$

which in the present instance has the appearance

$$\frac{S}{I} = \frac{2\mathcal{E}_s \left| \int_{-\infty}^{\infty} w^*(t; \rho_o, f_o) m(t; \rho_o, f_o) dt \right|^2}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1; \rho_o, f_o) \mathcal{K}(t_1, t_2) w(t_2; \rho_o, f_o) dt_1 dt_2} \quad (4.33)$$

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### D. THE OPTIMUM LINEAR PROCESSOR

The processors of major interest in this research are those which lead to a maximum signal-to-interference ratio at their output. These processors are called "optimum." More specifically, the optimum processor is characterized by the weighting function  $w_{\text{opt}}(t; \rho_0, f_0)$  which, among all weighting functions, yields the greatest value for  $\frac{S}{I}$  defined by equation (4.33).

Observe that the optimum weight function depends, of necessity, upon both the form of the echo to be detected  $m(t; \rho_0, f_0)$ , and the covariance function  $K(t_1, t_2)$  for the total received interference, since both of these terms appear explicitly in equation (4.33). Equations which determine the optimum processor will be presented shortly.

On occasion, however, reference will also be made in this research to another kind of linear processor, i.e. the processor which is optimum only if the received interference consists of white noise alone, without any clutter. The processor which is optimal under these restricted circumstances (but which is sub-optimal for the general problem being considered in this research) is called a "matched filter" or "simple matched filter" (or processor).

This terminology, chosen for its brevity and consistency with at least part of the literature<sup>†</sup>, is adhered to throughout this dissertation.

---

<sup>†</sup> The word "optimum" is almost universal in referring to a problem solution which maximizes some well defined criterion of merit. The phrase "matched filter" often enough has exactly the meaning given in the text, even though the interference is non-white. Reference 46 is an exact illustration of this latter usage.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### 1. Formal Solution for the General Case

It may be shown, as in Appendix A, that the processor weight function  $w(t; \rho_0, f_0)$  which maximizes  $\frac{S}{I}$  of equation (4.33) must satisfy

$$\int_{-\infty}^{\infty} K(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 = m(t_1; \rho_0, f_0) \quad (4.34)$$

for all  $t_1$ .

A basic analytical problem in this research is, therefore, the solution of the preceding integral equation for  $w(t_2; \rho_0, f_0)$ , once  $m(t_1; \rho_0, f_0)$  and  $K(t_1, t_2)$  have been specified. Means of generating such solutions will be discussed later, in chapters six, seven, and eight.

Here it will only be noted that a formal solution to the preceding integral equation may be written in the form

$$w(t_2; \rho_0, f_0) = \int_{-\infty}^{\infty} \mathcal{L}(t_2, t_3) m(t_3; \rho_0, f_0) dt_3, \quad (4.35)$$

which represents  $w(t; \rho_0, f_0)$  as some linear transformation  $\mathcal{L}$  of the modulation function  $m(t; \rho_0, f_0)$ . As shown in Appendix A, the kernel  $\mathcal{L}(t_2, t_3)$  introduced here must satisfy the equation

$$\int_{-\infty}^{\infty} K(t_1, t_2) \mathcal{L}(t_2, t_3) dt_2 = \sigma(t_1, t_3) \quad (4.36)$$

where  $\sigma(t_1, t_3)$  is a kernel with the "sifting property" for  $m(t; \rho_0, f_0)$ . That is

$$\int_{-\infty}^{\infty} \sigma(t_1, t_3) m(t_3; \rho_0, f_0) dt_3 = m(t_1; \rho_0, f_0) \quad (4.37)$$

for all  $t_1$ .

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Although this solution provides no indication of how to discover those kernels  $\sigma(t_1, t_2)$  and  $\mathcal{L}(t_1, t_2)$  which are appropriate for any particular case, it has the merit of providing general expressions for optimum system performance. Thus, in Appendix A it is shown that if  $w_{\text{opt}}(t_2; \rho_0, f_0)$  satisfies equation (4.34), then the corresponding (maximum) value of  $\frac{S}{I}$  is

$$\frac{S}{I} = 2 \mathcal{E}_{s-\infty} \int_{-\infty}^{\infty} w_{\text{opt}}^*(t_2; \rho_0, f_0) m(t_2; \rho_0, f_0) dt_2 \quad (4.38)$$

Once an expression for the optimum weight function is available, this equation provides a convenient calculation for the resulting optimum system performance. It, or its equivalent, is used in later chapters for this purpose.

One particular expression, equivalent to equation (4.38), has a certain suggestiveness for establishing the theoretical result of Chapter IX. It is obtained by noting that, since  $\sigma(t_1, t_3)$  defined by equation (4.37) has the role of an identity transformation, the solution of equation (4.36) for  $\mathcal{L}(t_2, t_3)$  amounts to the discovery of a kernel which is inverse to  $\mathcal{K}(t_1, t_2)$ . If one introduces the notation which is suggestive of this viewpoint by defining

$$\mathcal{K}^{-1}(t_2, t_3) = \mathcal{L}(t_2, t_3) \quad (4.39)$$

then equation (4.38) becomes

$$w_{\text{opt}}(t_2; \rho_0, f_0) = \int_{-\infty}^{\infty} \mathcal{K}^{-1}(t_2, t_3) m(t_3; \rho_0, f_0) dt_3 \quad (4.40)$$



and equation (4.38) becomes <sup>†</sup>

$$\frac{S}{I} = 2\mathcal{E}_s \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m^*(t_2; \rho_0, f_0) K^{-1}(t_2, t_3) m(t_3; \rho_0, f_0) dt_2 dt_3. \quad (4.41)$$

It is this last equation which has relevance for the bound to be derived in Chapter IX.

## 2. Solution for Uniformly Extended Clutter

The clutter problem most frequently analyzed arises out of the assumption, made either explicitly or implicitly, <sup>‡</sup> that the clutter dispersion function  $\mathcal{E}(\rho, f)$  may be written

$$\mathcal{E}(\rho, f) = \mathcal{E}_c \cdot Q(f) \quad (4.42)$$

where 
$$\int_{-\infty}^{\infty} Q(f) df = 1. \quad (4.43)$$

This corresponds to clutter interference extending over all range delays with a uniform mean power. The parameter  $\mathcal{E}_c$  is essentially a measure of the spatial density of the distributed radar cross-section of the clutter source. It has the dimensions of energy per unit range delay.

When Eqs. (4.42) and (4.18) are combined the clutter covariance function simplifies to

$$K_c(t_1, t_2) = \mathcal{E}_c \cdot 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(t_1 - \rho) m^*(t_2 - \rho) \cdot Q(f) \exp \{j2\pi f(t_1 - t_2)\} d\rho df. \quad (4.44)$$

<sup>†</sup> See Appendix A for the derivation.

<sup>‡</sup> This assumption underlies the analyses reported by George,<sup>14</sup> and Westerfield, et al.,<sup>46</sup> for examples.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The important fact is that now  $K_C(t_1, t_2)$  depends only upon the time difference  $t_1 - t_2$ , so it is possible to write

$$K_C(t_1, t_2) = K_C(\tau) = 2 E_C M(\tau) \cdot Q(\tau) \quad (4.45)$$

where

$$M(\tau) = \int_{-\infty}^{\infty} m(t_1 - \rho) m^*(t_2 - \rho) d\rho \quad (4.46)$$

$$Q(\tau) = \int_{-\infty}^{\infty} Q(f) \exp \{j2\pi f\tau\} df \quad (4.47)$$

$$\tau = t_1 - t_2$$

Under these circumstances the signal-to-interference ratio becomes

$$\frac{S}{I} = \frac{2 E_S \left| \int_{-\infty}^{\infty} w^*(t; \rho_0, f_0) m(t; \rho_0, f_0) dt \right|^2}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1; \rho_0, f_0) K(t_1 - t_2) w(t_2; \rho_0, f_0) dt_1 dt_2} \quad (4.48)$$

where the total interference covariance function  $K(t_1 - t_2)$  is the sum of clutter and noise covariance functions,

$$K(t_1 - t_2) = K_C(t_1 - t_2) + K_N(t_1 - t_2). \quad (4.49)$$

Because the function  $K(t_1 - t_2)$  depends only upon time difference, the problem of maximizing  $\frac{S}{I}$  given by (4.48) is formally identical to the problem of optimal receiver design for colored noise interference. In this latter context of noise alone an expression of the same outward form as (4.48) appeared as long ago as 1947.<sup>17</sup> The interpretation of the several functions appearing in equation (4.48) is, however, quite different for the present case and leads to substantial problems in the analysis. These problems will be discussed as they occur.

The solution for this case is conveniently carried out in the frequency domain. As shown in Appendix B the results are

$$W^*(f; \rho_o, f_o) = \frac{M^*(f; \rho_o, f_o)}{K(f)} = H(f; \rho_o, f_o) \quad (4.50)$$

and

$$\frac{S}{I} = 2 \mathcal{E}_s \int_{-\infty}^{+\infty} \frac{|M(f; \rho_o, f_o)|^2}{K(f)} df \quad (4.51)$$

where

$$W(f; \rho_o, f_o) = \mathcal{F}\{w(t; \rho_o, f_o)\} \quad (4.52)$$

$$M(f; \rho_o, f_o) = \mathcal{F}\{m(t; \rho_o, f_o)\} \quad (4.53)$$

$$K(f) = \mathcal{F}\{K(\tau)\} \quad (4.54)$$

The frequency response function  $H(f; \rho_o, f_o)$  is given in terms of the voltage spectrum of the desired echo  $M^*(f; \rho_o, f_o)$  divided by the power spectrum  $K(f)$  of the interference.

In the present case, however,  $K(f)$  is given by

$$K(f) = K_c(f) + K_n(f) \quad (4.55)$$

where

$$K_c(f) = \mathcal{F}\{K_c(\tau)\} \quad (4.56)$$

$K_n(f)$  = noise power spectral density function.

It is the fact that the interference spectrum  $K(f)$  depends upon the transmitted waveform (through Eqs. 4.45, 4.56, and 4.55) that distinguishes the present case from the case of colored noise. The interpretation of equations (4.50) and (4.51), for example, is made more difficult because  $K(f)$  and

$M(f; \rho_0, f_0)$  are related. In the present case it is not as if a signal were being received through colored noise of fixed spectrum. Rather the signal is being received in interference with a spectrum depending upon the signal. The effects, for example, of a change in the transmitted spectrum upon the signal-to-interference ratio cannot, therefore, be easily assessed in the present case. The very practical problem of choosing a "good" waveform for transmission is, therefore, rendered much more difficult.

This solution for uniformly extended clutter has been discussed first because of the prevalence of its basic assumption, equation (4.42), and the ease with which the solution is obtained. The simplicity, however, is achieved at the expense of possible variation of the clutter source with range delay,  $\rho$ . The assumed lack of clutter variation with range delay is never strictly true in a practical environment and possibly can be exploited for clutter rejection.

The problem of solving equation (4.34) in circumstances of more general clutter source distributions will be taken up again in Chapter VI. First, however, Chapter V will contain a description of the appropriate ambiguity function for describing delay and doppler performance of an optimum system.

V. AMBIGUITY FUNCTIONS

The viewpoint of the preceding chapter was strictly confined to the problem of designing a system for detecting an echo with some definite range delay,  $\rho_0$ , and some specified frequency shift,  $f_0$ . In the usual radar situation one is almost always also interested in an allied question. Assuming that a system has been designed to detect a specific signal, what is its response to other signals which might appear at its input? As was first observed by Woodward,<sup>49</sup> the answer to this question is contained in an appropriately defined "ambiguity function."

A. A GENERAL AMBIGUITY FUNCTION

Let it be supposed, at first quite generally, that the system under consideration has been designed to detect the echo  $r(t; \rho_0, f_0)$  by means of the weight function  $w(t; \rho_0, f_0)$ . Let  $u(\tau, \phi)$  denote the response to the echo  $r(t; \rho_0 + \tau, f_0 + \phi)$ . Then Eq. (4.6) gives  $u(\tau, \phi)$  as

$$u(\tau, \phi) = \int_{-\infty}^{\infty} w^*(t; \rho_0, f_0) r(t; \rho_0 + \tau, f_0 + \phi) dt. \quad (5.1)$$

This may be simplified somewhat by recalling that  $r(t; \rho_0 + \tau, f_0 + \phi)$  is the complex envelope of  $R(t; \rho_0 + \tau, f_0 + \phi)$  defined at Eq. (4.20). One can therefore write

$$\begin{aligned} r(t; \rho_0 + \tau, f_0 + \phi) &= A \cdot S(t - \rho_0 - \tau) \exp\{j2\pi(f_0 + \phi)(t - \rho_0 - \tau)\} \cdot \exp\{-j2\pi f_c t\} \\ &= A \cdot m(t - \rho_0 - \tau) \exp\{j2\pi f_c(t - \rho_0 - \tau)\} \\ &\quad \cdot \exp\{j2\pi(f_0 + \phi)(t - \rho_0 - \tau)\} \cdot \exp\{-j2\pi f_c t\} \\ &= A \cdot m(t - \rho_0 - \tau) \exp\{j2\pi(f_0 + \phi)(t - \rho_0 - \tau)\} \\ &\quad \cdot \exp\{-j2\pi f_c(\rho_0 + \tau)\}. \end{aligned} \quad (5.2)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

In consequence one may rewrite Eq. (5.1) as

$$u(\tau, \phi) = \Lambda \cdot \xi(\tau, \phi) \cdot \exp \{-j2\pi f_c(\rho_o + \tau)\} \quad (5.3)$$

$$\text{where} \quad \xi(\tau, \phi) = \int_{-\infty}^{\infty} w^*(t; \rho_o, f_o) m(t; \rho_o + \tau, f_o + \phi) dt \quad (5.4)$$

It will be shown that the function  $\xi(\tau, \phi)$  is, in appropriate circumstances, essentially equivalent to Woodward's<sup>49</sup> ambiguity function  $\psi(\tau, \phi)$  and its generalizations.<sup>37,41</sup> In the form given above however,  $\xi(\tau, \phi)$  is defined for an arbitrary, neither necessarily "optimal" nor "matched," weight function. It therefore describes the response of an arbitrary linear system to a class of input signals. Depending upon the system, it may also be a function of  $\rho_o$  and  $f_o$  as well as  $\tau$  and  $\phi$ .

The general ambiguity function  $\xi(\tau, \phi)$  may be used to re-express Eq. (4.31) in a more suggestive form. Equation (4.18) is introduced into (4.31) to yield

$$\begin{aligned} \left\langle |c(\rho_o, f_o)|^2 \right\rangle &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1; \rho_o, f_o) w(t_2; \rho_o, f_o) \\ &\quad \cdot 2\mathcal{E}(\rho, f) m(t_1; \rho, f) m^*(t_2; \rho, f) dt_1 dt_2 d\rho df. \end{aligned} \quad (5.5)$$

When the definition (5.4) is employed, the result for the mean square output clutter interference is

$$\left\langle |c(\rho_o, f_o)|^2 \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\mathcal{E}(\rho_o + \tau, f_o + \phi) \cdot |\xi(\tau, \phi)|^2 d\tau d\phi \quad (5.6)$$

wherein the following changes of variable have been made

$$\rho = \rho_o + \tau \quad (5.7a)$$

and

$$f = f_o + \phi \quad (5.7b)$$

The signal-to-interference ratio itself may be expressed in terms of the general ambiguity function, although this seems more useful for interpretation than for optimization of system performance in clutter. If the processor weight function is assumed to be normalized so that†

$$\int_{-\infty}^{\infty} w^*(t; \rho_0, f_0) w(t; \rho_0, f_0) dt = 1 \quad (5.8)$$

and the noise interference is assumed "white," then one can rewrite Eq. (4.33) to yield

$$\frac{S}{I} = \frac{2\mathcal{E}_s |\xi(0,0)|^2}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2\mathcal{E}(\rho_0 + \tau, f_0 + \phi) |\xi(\tau, \phi)|^2 d\tau d\phi + N_0} \quad (5.9)$$

With both the clutter dispersion function  $\mathcal{E}(\rho, f)$  and the general ambiguity function  $\xi(\tau, \phi)$  in full view in Eq. (5.9) it is tempting to conclude that one might

- i) find a  $\xi(\tau, \phi)$  such that  $\xi(0,0)$  is relatively large; and
- ii) find a  $\xi(\tau, \phi)$  which minimizes the contribution of the double integral in the denominator.

A great difficulty in attempting this intuitively appealing approach to maximizing  $\frac{S}{I}$  is the lack of a sufficient characterization for an ambiguity function. Thus while one might choose, or derive, a  $\xi(\tau, \phi)$  satisfying the preceding two conditions, there will not necessarily exist an  $m(t; \rho, f)$  and  $w^*(t; \rho_0, f_0)$  from which the desired  $\xi(\tau, \phi)$  can be derived by Eq. (5.4).

† This is not necessarily the same normalization used to write (4.38).

Because of this difficulty, attention in this research has been restricted to maximization of  $\frac{S}{I}$  in the analytically more tractable form given at (4.33). The drawback to the latter approach, however, is that the ambiguity function which results after the optimal weight function is found is not easily or directly controllable.

#### B. UNIT TOTAL AMBIGUITY

One particular property of the general ambiguity function which has proven useful in this research, however, is the property of having "unit total ambiguity." Suppose that both  $w(t; \rho_0, f_0)$  and  $m(t; \rho_0, f_0)$  are normalized, square-integrable functions; that is

$$\int_{-\infty}^{\infty} |w(t; \rho_0, f_0)|^2 dt = 1 \quad (5.10)$$

for any  $(\rho_0, f_0)$ , and

$$\int_{-\infty}^{\infty} |m(t; \rho_0, f_0)|^2 dt = 1 \quad (5.11)$$

for any  $(\rho_0, f_0)$ . Then it may be shown<sup>†</sup> that the general ambiguity function  $\xi(\tau, \phi)$ , already defined by

$$\xi(\tau, \phi) = \int_{-\infty}^{\infty} w^*(t; \rho_0, f_0) m(t; \rho_0 + \tau, f_0 + \phi) dt, \quad (5.12)$$

has the property that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\xi(\tau, \phi)|^2 d\tau d\phi = 1. \quad (5.13)$$

<sup>†</sup> See equations ( 5 ), ( 8 ), and ( 9 ) of reference 41 .



Two particular aspects of this useful result should be noticed. In the first place,  $\xi(\tau, \phi)$  is defined by equation (5.12) in terms of a single, fixed  $w(t; \rho_0, f_0)$  which need only exist for the parameter set  $(\rho_0, f_0)$ . This can be a convenience in situations where properties for other parameter sets might be uncertain.

In the second place, the function  $w(t; \rho_0, f_0)$  appearing in equation (5.12) is, except for equation (5.10), essentially unconstrained. Thus, despite the suggestiveness of the notation, it need be neither an optimum weight function nor a matched weight function, for example. An arbitrary, square-integrable function of  $t$  will therefore serve for  $w(t; \rho_0, f_0)$  in equations (5.10) and (5.12).

#### C. AMBIGUITY FUNCTIONS FOR STATIONARY SYSTEMS

When the interfering process is statistically stationary, it is a simple matter to show that the optimal processor is likewise stationary. That is, optimal processors for two signals which differ only by a time translation will themselves only differ by the corresponding time translation. Thus, if  $w(t; \rho_0, f_0)$  satisfies

$$\int_{-\infty}^{\infty} \mathcal{K}(t_1 - t_2) w(t_2; \rho_0, f_0) dt_2 = m(t_1; \rho_0, f_0) \quad (5.14)$$

then it should be clear that  $w(t_2 - \tau; \rho_0, f_0)$  satisfies

$$\int_{-\infty}^{\infty} \mathcal{K}(t_1 - t_2) w(t_2 - \tau; \rho_0, f_0) dt_2 = m(t_1 - \tau; \rho_0, f_0) . \quad (5.15)$$

Moreover, since  $m(t; \rho_0, f_0)$  as defined at Eq. (4.17) has the property that

$$m(t; \rho_0, f_0) \equiv m(t - \rho_0; 0, f_0) \quad (5.16)$$

it follows that the optimal processor satisfying (5.14) will likewise have the property that

$$w(t; \rho_0, f_0) \equiv w(t - \rho_0; 0, f_0) . \quad (5.17)$$

Using now Eqs. (5.16) and (5.17), it is possible to simplify the general ambiguity function of Eq. (5.4) for the present case. From (5.4)

$$\begin{aligned} \xi(\tau, \phi) &= \int_{-\infty}^{\infty} w^*(t; \rho_0, f_0) m(t; \rho_0 + \tau, f_0 + \phi) dt \\ &= \int_{-\infty}^{\infty} w^*(t - \rho_0; 0, f_0) m(t - \rho_0 - \tau; 0, f_0 + \phi) dt \\ &= \int_{-\infty}^{\infty} w^*(t; 0, f_0) m(t - \tau; 0, f_0 + \phi) dt , \end{aligned} \quad (5.18)$$

wherein it is noted that the possible dependence of  $\xi(\tau, \phi)$  upon  $\rho_0$  has been removed. When the explicit definition (4.17) of  $m(t; \rho, f)$  is introduced into (5.15) the result for  $\xi(\tau, \phi)$  is

$$\xi(\tau, \phi) = \int_{-\infty}^{\infty} w^*(t; 0, f_0) m(t - \tau) \exp \{ j 2 \pi (f_0 + \phi) (t - \tau) \} dt . \quad (5.19)$$

Parseval's theorem in turn yields the alternative expression

$$\xi(\tau, \phi) = \int_{-\infty}^{\infty} W^*(f; 0, f_0) M(f - f_0 - \phi) \exp \{ j 2 \pi f \tau \} df \quad (5.20)$$

where

$$W(f; 0, f_0) = \mathcal{F} \{ w(t; 0, f_0) \} \quad (5.21a)$$

$$M(f) = \mathcal{F} \{ m(t) \} . \quad (5.21b)$$

Finally Eq. (4.50) for the optimal processor in the frequency domain, together with (5.20), yields

$$\xi(\tau, \phi) = \int_{-\infty}^{\infty} \frac{M^*(f-f_0)M(f-f_0-\phi)}{K(f)} \exp \{j2\pi f\tau\} df. \quad (5.22)$$

This is the expression for the ambiguity function of the linear processor which is optimal for detection in colored, statistically stationary interference. Note first, that  $\xi(\tau, \phi)$  is not independent of  $f_0$  in general; and second, that for interference which includes clutter,  $K(f)$  depends upon  $M(f)$ .

In the case of stationary "white" interference, when  $K(f)$  is constant, the ambiguity function for the optimal processor (now a "matched filter") becomes

$$\xi(\tau, \phi) = \int_{-\infty}^{\infty} M^*(f)M(f-\phi) \exp \{j2\pi f\tau\} df \cdot \exp \{-j2\pi f_0\tau\} \quad (5.23)$$

agreeing essentially with Woodward's original definition.<sup>49</sup>

VI. GENERAL FORMS OF SOLUTION

This and the next two chapters will be concerned with the problem of solving the basic integral equation presented in Chapter III. In certain respects the integral equation resembles a conventional Fredholm integral equation of the second kind and existing theory can be used in its solution. In other respects, however, the integral equation has a structure essentially different from the conventional Fredholm equation and its solution poses problems not heretofore considered in the engineering literature.

Following a discussion of the integral equation itself, this chapter concludes with the presentation of three different forms for the general solution.

A. DISCUSSION OF THE INTEGRAL EQUATION

This research is limited to the consideration of interference consisting of clutter plus white noise. The interference covariance function is therefore given by

$$\mathcal{K}(t_1, t_2) = \mathcal{K}_c(t_1, t_2) + N_0 \cdot \delta(t_2 - t_1) \quad (6.1)$$

where  $\mathcal{K}_c(t_1, t_2)$  is the clutter covariance function already given at Eq. (4.18), and  $N_0$  is the noise power spectral density in watts per cycle per sec. The introduction of this form of interference kernel into Eq. (4.34) yields, for the integral equation which must be solved,

$$\int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 + N_0 \cdot w(t_1; \rho_0, f_0) = m(t_1; \rho_0, f_0) \quad (6.2)$$

where

$$\mathcal{K}_c(t_1, t_2) = 2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(t_1; \rho, f) m^*(t_2; \rho, f) \mathcal{G}(\rho, f) d\rho df. \quad (6.3)$$

If one's attention is for the moment confined to just Eq. (6.2), then one sees an equation having the form of the Fredholm linear integral equation of the second kind. For such an equation general forms of solution do exist and will presently be exhibited. The utility and interpretation of these forms, however, often depends upon the linearity of the relation which exists between the given (or "forcing") function  $m(t; \rho_0, f_0)$  and the solution (or "response") function  $w(t; \rho_0, f_0)$ , for any fixed kernel  $\mathcal{K}_c(t_1, t_2)$ .

When one's view is broadened, as it must for the present problem, to include the kernel  $\mathcal{K}_c(t_1, t_2)$  defined by Eq. (6.3), then a considerably different situation arises. In the first place, the kernel  $\mathcal{K}_c(t_1, t_2)$  may not be regarded as a fixed function specified independently of  $m(t; \rho_0, f_0)$ .† This certainly introduces technical difficulties into the analysis.

Of possibly greater significance, however, is the fact that linearity no longer exists between  $m(t; \rho_0, f_0)$  and the solution  $w(t; \rho_0, f_0)$ . Thus one cannot in general expect to write the solution as a superposition of the separate responses to elemental functions which constitute  $m(t; \rho_0, f_0)$ .

One can indeed see, virtually by inspection, that if  $w(t; \rho_0, f_0)$  is the solution for  $m(t; \rho_0, f_0)$ , then the solution for  $a \cdot m(t; \rho_0, f_0)$  is certainly not  $a \cdot w(t; \rho_0, f_0)$ , for any

---

† One consequence of this fact has already been noted in connection with the solution for uniformly extended clutter, Sec III.D.2.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

non-zero scalar "a." Likewise, if  $w_1(t; \rho_o, f_o)$  and  $w_2(t; \rho_o, f_o)$  are the respective solutions for some  $m_1(t; \rho_o, f_o)$  and  $m_2(t; \rho_o, f_o)$ , then the solution for

$$m(t; \rho_o, f_o) = m_1(t; \rho_o, f_o) + m_2(t; \rho_o, f_o) \quad (6.4)$$

is not

$$w(t; \rho_o, f_o) = w_1(t; \rho_o, f_o) + w_2(t; \rho_o, f_o) \quad (6.5)$$

The techniques of linear analysis therefore have only a restricted applicability in the present research.

It will be seen† that, as a further consequence of the particular structure of Eqs. (6.2) and (6.3), the situation tends not to be one where a single solution is applicable to a collection of cases of interest. Rather, different cases tend to present essentially different problems which are susceptible to different modes of solution, if they are solvable at all.

Similar comments apply with respect to the lack of any linear influence of the clutter energy dispersion function  $\mathcal{E}(\rho, f)$  upon the problem solution. It will in fact be seen that one parameter which greatly influences the form of the solution is the clutter-to-noise power ratio. This, however, is not entirely unexpected, since this ratio directly determines the relative importance of the two terms on the left of Eq. (6.2).

In physical terms appropriate to the present research, these considerations imply that the optimum data processor depends in an unobvious manner upon not only the detailed

---

† In Chapters X through XIII, for example.

functional forms of the transmitter modulation and the assumed clutter source, but also upon the relative levels of clutter and noise interference. In particular, since only the clutter component of interference depends upon the transmitter power, the optimum data processor also depends upon transmitter power.

All these considerations lead to the conclusion that the integral equation defined by both Eqs. (6.2) and (6.3) is essentially new to the literature and possesses solutions with characteristics only imperfectly understood.

#### B. SOLUTION IN TERMS OF EIGENFUNCTIONS

In this section the Schmidt-Hilbert method is applied to Eq. (6.2) to yield a general form for the solution. The method is applicable because the kernel  $\mathcal{K}_c(t_1, t_2)$  is Hermitian and, in a practical situation, square-integrable over the plane.†

That the kernel is Hermitian is verified by direct inspection of Eq. (6.3). Since  $\mathcal{E}(\rho, f)$  is a real-valued function one can see that, as required,

$$\mathcal{K}_c(t_1, t_2) \equiv \mathcal{K}_c^*(t_2, t_1) \quad (6.6)$$

The second requirement, that the kernel be square integrable, is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\mathcal{K}_c(t_1, t_2)|^2 dt_1 dt_2 < \infty \quad (6.7)$$

As will now be seen, this integral will be bounded for suitably bounded dispersion functions  $\mathcal{E}(\rho, f)$ . Using the definition (6.3) one can write directly

---

† See p. 242, Riesz and Nagy.<sup>36</sup>

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | \mathcal{K}_c(t_1, t_2) |^2 dt_1 dt_2 \equiv \\
 & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2 \mathcal{E}(\rho, f) m(t_1; \rho, f) m^*(t_2; \rho, f) d\rho df \right\} \cdot \\
 & \cdot \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2 \mathcal{E}(\rho', f') m^*(t_1; \rho', f') m(t_2; \rho', f') d\rho' df' \right\} dt_1 dt_2 \\
 & \hspace{15em} (6.8)
 \end{aligned}$$

This is condensed considerably by introducing the function

$$g(\rho', f'; \rho, f) \equiv \int_{-\infty}^{\infty} m(t_1; \rho, f) m^*(t_1; \rho', f') dt_1, \quad (6.9)$$

and introducing it into Eq. (6.8), after changing the order of integration. The result is

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | \mathcal{K}_c(t_1, t_2) |^2 dt_1 dt_2 \equiv \\
 & 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}(\rho, f) \mathcal{E}(\rho', f') | g(\rho', f'; \rho, f) |^2 d\rho df d\rho' df' \\
 & \hspace{15em} (6.10)
 \end{aligned}$$

It has been assumed, however, that for any  $(\rho, f)$

$$\int_{-\infty}^{\infty} | m(t_1; \rho, f) |^2 dt_1 = 1 \quad (6.11)$$

Since Eq. (6.9) therefore has the same form as Eq. (5.12), one concludes from Eq. (5.13) that, for any  $(\rho, f)$ ,



$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(\rho', f'; \rho, f)|^2 d\rho' df' = 1. \quad (6.12)$$

Integrating first with respect to the primed variables in Eq. (6.10) therefore yields

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |H_c(t_1, t_2)|^2 dt_1 dt_2 \leq 4 \hat{G}_c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\rho, f) d\rho df \quad (6.13)$$

where  $G(\rho, f)$  has been assumed to be a continuous (everywhere) function of  $\rho$  and  $f$ , and where

$$\hat{G}_c = \max_{(\rho, f)} G(\rho, f). \quad (6.14)$$

The inequality (6.13) stems from the "allotment" by Eq. (6.12) of unit "volume" under the surface  $|g(\rho', f'; \rho, f)|^2$ . The maximum value for the integral in Eq. (6.10) is therefore achieved if  $|g(\rho', f'; \rho, f)|$  is such that its content is concentrated at the value of  $(\rho', f')$  where  $G(\rho, f)$  has its maximum value  $\hat{G}_c$ . Distribution of the fixed volume in any other manner leads to lesser values for the integral.

A further practical matter, which acts only to strengthen the inequality (6.13), should be noted in passing. The function  $g(\rho', f'; \rho, f)$  is essentially the auto-ambiguity function for the waveform  $m(t)$ . As such it cannot be specified arbitrarily,† and strict equality in (6.13) is not necessarily to be expected for any particular  $m(t)$ . From the inequality (6.13), one therefore concludes that, if

$$\hat{G}_c \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\rho, f) d\rho df < \infty \quad (6.15)$$

---

† Constraints upon the auto-ambiguity function are given by Price and Hofstetter,<sup>34</sup> and Westerfield, et al.,<sup>46</sup>

then

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\mathcal{K}_c(t_1, t_2)|^2 dt_1 dt_2 < \infty \quad (6.16)$$

as required.

The condition (6.15) is unrestrictive in practical cases. Boundedness of the double integral, for example, simply requires that the total energy returned from the clutter source be finite. A finite value for  $\hat{\mathcal{E}}_c$  is likewise readily assumed.<sup>†</sup>

Assuming that the Hermitian kernel  $\mathcal{K}_c(t_1, t_2)$  does satisfy the inequality (6.13), it then has a finite or countable set of eigenfunctions.<sup>3\*</sup> These eigenfunctions, denoted  $\phi_j(t)$ , and their corresponding eigenvalues  $\mu_j$  may be characterized, with no loss generality, as having the following properties

$$i) \mathcal{K}_c \phi_j = \mu_j \cdot \phi_j \quad j = 1, 2, 3, \dots \quad (6.17a)$$

$$ii) (\phi_j, \phi_j) = 1 \quad j = 1, 2, 3, \dots \quad (6.17b)$$

$$iii) (\phi_j, \phi_k) = 0 \quad \text{for all different } j \text{ and } k \neq 0 \quad (6.17c)$$

where the following conventional notations for linear transformations and scalar products have been introduced for brevity:

$$\mathcal{K}_c f \equiv \int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) f(t_2) dt_2 \quad (6.17d)$$

<sup>†</sup> Cases where  $\mathcal{E}(\rho, f)$  is impulsive in one variable or the other, as for stationary clutter, can be handled separately and shown to yield a similar result.

and

$$(f, g) \equiv \int_{-\infty}^{\infty} f^*(t)g(t)dt \quad . \quad (6.17e)$$

The possibility of having zero for an eigenvalue of  $\mathcal{K}_c(t_1, t_2)$  is, however, not yet excluded. Until and unless it is, for the particular functions  $m(t; \rho, f)$  and  $\mathcal{G}(\rho, \varepsilon)$  appearing in the case of interest, the analysis must include the possibility of square-integrable functions, not identically zero, with the properties

$$\text{iv) } \mathcal{K}_c \phi_o = 0 \quad (6.18a)$$

$$\text{v) } (\phi_j, \phi_o) = 0 \quad j = 1, 2, \dots \quad (6.18b)$$

for a typical such function  $\phi_o(t)$ .

#### 1. The Schmidt-Hilbert Solution

With these preliminary observations concluded, the solution of Eq. (6.2) becomes a straightforward matter. For the desired echo to be detected, namely  $m(t; \rho_o, \varepsilon_o)$ , form the coefficients  $\alpha_j$  defined by

$$\alpha_j = (\phi_j, m) \quad (6.19)$$

and then consider the function  $m_o(t)$  defined by

$$m_o(t) = m(t; \rho_o, \varepsilon_o) - \sum_{j=1}^{\infty} \alpha_j \phi_j(t) \quad . \quad (6.20)$$

Using Eqs. (6.17), (6.19), and (6.20), one may verify that

$$(\phi_j, m_o) = 0 \quad j = 1, 2, \dots \quad (6.21)$$

Moreover,† for  $m_o(t)$  defined as in Eq. (6.20),

$$\mathcal{H}_c m_o = 0 \quad (6.22)$$

The desired echo is, in this manner, represented by

$$m(t; \rho_o, f_o) = m_o(t) + \sum_{j=1}^{\infty} \alpha_j \phi_j(t) \quad (6.23)$$

Let it be assumed that the solution of Eq. (6.2) has the similar representation

$$w(t; \rho_o, f_o) = w_o(t) + \sum_{j=1}^{\infty} \beta_j \phi_j(t) \quad (6.24)$$

where the  $\beta_j$  and  $w_o(t)$  are to be determined, and

$$\mathcal{H}_c w_o = 0. \quad (6.25)$$

Combination of Eqs. (6.17), (6.23), (6.24), and (6.25) with Eq. (6.2) then yields

$$\sum_{j=1}^{\infty} \beta_j \mu_j \phi_j(t) + N_c \left\{ w_o(t) + \sum_{j=1}^{\infty} \beta_j \phi_j(t) \right\} = m_o(t) + \sum_{j=1}^{\infty} \alpha_j \phi_j(t) \quad (6.26)$$

Upon forming the scalar product of each side of this equation with  $\phi_k(t)$ , and then using the properties (6.17b), (6.17c), and (6.18a), one finds the requirement that, for  $k = 1, 2, \dots$ ,

$$\beta_k \cdot \mu_k + N_o \cdot \beta_k = \alpha_k \quad (6.27)$$

---

† cf. Theorem on p. 242, Riesz-Nagy.

or

$$\beta_k = \frac{\alpha_k}{\mu_k + N_o} \quad (6.28)$$

Moreover, one concludes that

$$w_o(t) = \frac{1}{N_o} \cdot m_o(t) \quad (6.29)$$

Incorporation of these results into Eq. (6.24) finally yields, for the optimum processor weight function,

$$w_{opt}(t; \rho_o, f_o) = \frac{1}{N_o} \cdot m_o(t) + \sum_{k=1}^{\infty} \frac{\alpha_k}{\mu_k + N_o} \cdot \phi_k(t) \quad (6.30)$$

The performance of the optimum processor is derived using the basic Eq. (4.38) together with (6.30) and (6.23). After simplification the result is

$$\left( \frac{S}{I} \right)_{opt} = 2 \mathcal{E}_s \left\{ \frac{(m_o, m_o)}{N_o} + \sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{\mu_k + N_o} \right\} \quad (6.31)$$

If the kernel  $\mathcal{K}_c(t_1, t_2)$  has no eigenvalues equal to zero, then the set of eigenfunctions  $\phi_k(t)$  form a complete, orthonormal set in the space of functions  $f(t)$  square integrable on the line  $-\infty < t < \infty$ .<sup>†</sup> Under these circumstances the functions  $m_o(t)$  and  $w_o(t)$  have zero norm. The preceding results may then be reduced to

$$w_{opt}(t; \rho_o, f_o) = \sum_{k=1}^{\infty} \frac{\alpha_k}{\mu_k + N_o} \phi_k(t) \quad (6.32)$$

---

<sup>†</sup> cf. theorem on p. 234 of Riesz-Nagy.<sup>36</sup>

and

$$\left(\frac{S}{I}\right)_{\text{opt}} = 2 \mathcal{E}_s \sum_{k=1}^{\infty} \frac{|\alpha_k|^2}{\mu_k + N_0} \quad (6.33)$$

The simplicity of these results is somewhat deceptive because the fact has not been explicitly indicated that both the eigenfunctions as well as their eigenvalues depend upon the modulation function  $m(t)$  and the clutter dispersion function  $\mathcal{E}(\rho, f)$ . Moreover there is no simple relationship between the various quantities, other than that provided by the defining equation (from 6.17)

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2 \mathcal{E}(\rho, f) m(t_1; \rho, f) m^*(t_2; \rho, f) d\rho df \right\} \phi_k(t_2) dt_2 = \mu_k \phi_k(t_1) \quad (6.34)$$

The first difficulty with this form of solution is therefore the discovery of the  $\mu_k$  and  $\phi_k(t)$  and their relation to  $m(t)$  and  $\mathcal{E}(\rho, f)$ . The second difficulty is relating changes in either  $m(t)$  or  $\mathcal{E}(\rho, f)$  to resulting changes in  $w_{\text{opt}}(t; \rho_0, f_0)$  and, ultimately, in  $\left(\frac{S}{I}\right)_{\text{opt}}$ . It is simply not clear what happens to the value of the sum in Eq. (6.33), for example, as all the  $\mu_k$  and  $\alpha_k$  change in response to a single change in, say,  $m(t)$ .

The solution for uniformly extended clutter which was presented in Eqs. (4.39) through (4.53) provides a direct illustration of these remarks because it is, after all, an eigenfunction solution. In fact, for a stationary covariance kernel  $\mathcal{K}_c(t_1 - t_2)$  the eigenfunctions are always known, since

$$\int_{-\infty}^{\infty} \mathcal{K}_c(t_1 - t_2) \cdot \exp \left\{ -j2\pi f t_2 \right\} dt_2 \equiv K_c(f) \cdot \exp \left\{ -j2\pi f t_1 \right\} \quad (6.35)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

where  $K_c(f) = \mathcal{F}\{ \mathcal{H}_c(\tau) \}$ .

In the case of uniformly extended clutter one therefore always has the identification

$$\phi_f(t) = \exp \{ -j2\pi ft \} \quad (6.36)$$

and

$$\mu_f = K_c(f) \quad (6.37)$$

Because the kernel  $\mathcal{H}_c(t_1 - t_2)$  is not bounded, in the sense of Eq. (6.7), the discrete index set  $k = 1, 2, \dots$  goes over into the continuous variable  $f$ , and the sums in Eqs. (6.30) through (6.33) are replaced by integrations with respect to  $f$ . Thus the analogue of Eq. (6.33) becomes

$$\left( \frac{S}{I} \right)_{\text{opt}} = 2 \mathcal{E}_s \int_{-\infty}^{\infty} \frac{|M(f; \rho_o, f_o)|^2}{K_c(f) + N_o} df, \quad (6.38)$$

exactly as seen earlier at Eq. (4.48).

The difficulty of relating system performance to waveform parameters, even for the very easily interpreted Eq. (6.38), has already been remarked at the conclusion of Chapter IV. One additional example here will suffice.

If  $\mathcal{H}_c(t_1 - t_2)$  were a fixed function independent of  $m(t)$ , as it is not, there would be a direct answer to the question: "What should be the waveform  $m(t)$ , in order to maximize  $(\frac{S}{I})$ ?" The general answer would be: "Let  $m(t)$  be exactly that eigenfunction which has the least eigenvalue," for then the sum in Eq. (6.33) would reduce to a single term with the greatest numerator (unity) and the least denominator.

In the present case it should be clear by now that no such direct answer is so easily available to that most important question of waveform design. The difficulty discussed at the end of Chapter IV reappears here in somewhat more general form. Because the kernel  $\mathcal{K}_c(t_1, t_2)$  in general depends upon  $m(t)$ , any alteration of  $m(t)$  leads to changes of not only the  $\alpha_k$  appearing in Eqs. (6.30) and (6.31), but also the basic eigenvectors  $\phi_k(t)$  and their eigenvalues  $\mu_k$ . Assessment of the influence of any change in  $m(t)$  upon either the processor weight function or system performance is therefore not to be had directly by inspection of this solution, Eqs. (6.30) or (6.31), for the general case.

## 2. Bounds for Eigenvalues

The eigenvalues  $\mu_k$ , defined by Eq. (6.17a) and appearing prominently in the preceding solution, may be bounded directly in terms of  $\mathcal{E}(\rho, f)$ .

Equation (6.17a), written in full, asserts that

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(t_1; \rho, f) m^*(t_2; \rho, f) 2 \mathcal{E}(\rho, f) d\rho df \right\} \phi_k(t_2) dt_2 = \mu_k \phi_k(t_1) . \quad (6.39)$$

If one defines the function

$$\xi_k(\rho, f) = \int_{-\infty}^{\infty} m^*(t_2; \rho, f) \phi_k(t_2) dt_2 \quad (6.40)$$

then, after changing the order of integration, Eq. (6.39) may be written

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(t_1; \rho, f) 2 \mathcal{E}(\rho, f) \xi_k(\rho, f) d\rho df = \mu_k \phi_k(t_1) . \quad (6.41)$$



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Multiplying both sides by  $\phi_k^*(t_1)$  and then integrating over the line  $-\infty < t_1 < \infty$ , yields

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi_k^*(\rho, f) \mathcal{E}(\rho, f) \xi_k(\rho, f) d\rho df = \mu_k \int_{-\infty}^{\infty} |\phi_k(t_1)|^2 dt_1 \quad (6.42)$$

or

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}(\rho, f) |\xi_k(\rho, f)|^2 d\rho df = \mu_k \quad (6.43)$$

But now, Eq. (5.13) may be used to conclude that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\xi_k(\rho, f)|^2 d\rho df = 1 \quad (6.44)$$

since both

$$\int_{-\infty}^{\infty} |\eta^*(t; \rho, f)|^2 dt = 1 \quad (6.45)$$

and

$$\int_{-\infty}^{\infty} |\phi_k(t)|^2 dt = 1 \quad (6.46)$$

as required by Eqs. (5.10) and (5.11).

One then concludes, by inspection of Eq. (6.43) that, since both  $\mathcal{E}(\rho, f)$  and  $|\xi_k(\rho, f)|^2$  are everywhere non-negative,

$$\mu_k \geq 0. \quad (6.47)$$

One further concludes that, for a fixed choice of function  $\mathcal{E}(\rho, f)$ , one can by no means cause the unit-volume allotted to  $|\xi_k(\rho, f)|^2$  by Eq. (6.44) to be disposed over the  $(\rho, f)$

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

plane in such a fashion as to cause the integral on the left of Eq. (6.43) to exceed the maximum value of  $2\mathcal{G}(\rho, f)$ . Thus, if  $2\mathcal{G}(\rho, f)$  is taken to be a continuous function of  $\rho$  and  $f$ , one may write

$$0 \leq \mu_k \leq 2 \cdot \max_{(\rho, f)} \mathcal{G}(\rho, f) . \quad (6.48)$$

The utility of these bounds in limiting achievable system performance will be seen in Chapter IX, once other form of problem solution have been considered.

### C. SOLUTION AS A RATIONAL FUNCTION OF CLUTTER-TO-NOISE RATIO

In general, the ratio of clutter power to noise power depends upon many things, such as the transmitted waveform, the detailed structure of the dispersion function  $\mathcal{G}(\rho, f)$ , and whether one is speaking of the interference ahead of, or following, the data processor. It will also be a time-variable ratio, in general. A single, simple parameter which indicated the general levels of clutter and noise interference might therefore be a convenience.

One such parameter has appeared in this research and is defined by

$$\mathcal{R}_c = \frac{\hat{2\mathcal{G}}_c}{N_o} \quad (6.49)$$

where

$$\hat{\mathcal{G}}_c = \max_{(\rho, f)} \mathcal{G}(\rho, f) . \quad (6.50)$$

Although  $\mathcal{R}_c$  does not specify the actual ratio of clutter and noise powers at any given time for any given location in the system as the result of any given waveform and dispersion function, it is nevertheless true that the actual ratios are

proportional to  $R_c$  once the shapes, but not the amplitudes, have been chosen for the modulation function  $m(t)$  and the dispersion function  $\mathcal{E}(\rho, f)$ .

The parameter  $R_c$  appears naturally in the following analysis.

### 1. The Fredholm Solution

The method due to Fredholm (see Riesz-Nagy, <sup>36</sup> pp. 172 ff.) may also be applied to the solution of Eq. (6.2), after a preliminary adjustment of notation.

Let the dispersion function  $\mathcal{E}(\rho, f)$  be factored as follows

$$\mathcal{E}(\rho, f) = \hat{\mathcal{E}}_c \cdot E(\rho, f) \quad (6.51)$$

with the result that  $\mathcal{K}_c(t_1, t_2)$  becomes

$$\mathcal{K}_c(t_1, t_2) = 2\hat{\mathcal{E}}_c \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(t_1, \rho, f) m^*(t_2, \rho, f) E(\rho, f) d\rho df \quad (6.52)$$

which may be written

$$\mathcal{K}_c(t_1, t_2) = 2\hat{\mathcal{E}}_c \cdot K_c(t_1, t_2) . \quad (6.53)$$

Assuming that  $N_0 \neq 0$ , one may now rewrite Eq. (6.2) in the form

$$w(t_1; \rho_0, f_0) + R_c \cdot \int_{-\infty}^{\infty} K_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 = \frac{1}{N_0} m(t_1; \rho_0, f_0) \quad (6.54)$$

Fredholm's method of solution first requires the recursive generation of kernels  $D_n(t_1, t_2)$ ,  $n = 1, 2, \dots$ , in the following manner.<sup>†</sup> The process begins with

$$D_0(t_1, t_2) = K_c(t_1, t_2) \quad (6.55)$$

and then, for  $n = 1, 2, \dots$ , generates  $D_n(t_1, t_2)$  according to

$$D_n(t_1, t_2) = K_c(t_1, t_2) \cdot D_n - n \int_{-\infty}^{\infty} K_c(t_1, \tau) D_{n-1}(\tau, t_2) d\tau. \quad (6.56)$$

where the coefficient  $D_n$  is always available from the preceding kernel, according to

$$D_n = \int_{-\infty}^{\infty} D_{n-1}(\tau, \tau) d\tau. \quad (6.57)$$

After definition of

$$D(\mathcal{R}_c) = \sum_{n=0}^{\infty} \frac{1}{n!} D_n \mathcal{R}_c^n \quad (6.58)$$

and

$$D(t_1, t_2; \mathcal{R}_c) = K_c(t_1, t_2) + \sum_{n=1}^{\infty} \frac{1}{n!} D_n(t_1, t_2) \cdot \mathcal{R}_c^n \quad (6.59)$$

the optimal processor may then be written

$$w_{\text{opt}}(t_1; \rho_o, f_o) = \frac{1}{N_o} m(t_1; \rho_o, f_o) - \mathcal{R}_c \int_{-\infty}^{\infty} \frac{D(t_1, t_2; \mathcal{R}_c)}{D(\mathcal{R}_c)} \frac{m(t_2; \rho_o, f_o)}{N_o} dt_2 \quad (6.60)$$

<sup>†</sup> The notations and presentations of Korn and Korn,<sup>22</sup> p. 436, and Margenau and Murphy,<sup>28</sup> pp. 526-7, form the basis for the description which is given.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

with optimum performance given by

$$\left(\frac{S}{I}\right)_{\text{opt}} = \frac{2\mathcal{E}_s}{N_0} - \frac{2\mathcal{E}_s}{N_0} \cdot \mathcal{R}_c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m^*(t_1; \rho_0, f_0) \frac{D(t_1, t_2; \mathcal{R}_c)}{D(\mathcal{R}_c)} m(t_2; \rho_0, f_0) dt_1 dt_2 \quad (6.61)$$

Subject only to the requirement<sup>†</sup> that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K_c(t_1, t_2)|^2 dt_1 dt_2$$

exist and be non-zero (cf. inequality 6.7), the power series for both  $D(\mathcal{R}_c)$  and  $D(t_1, t_2; \mathcal{R}_c)$  converge for all finite  $\mathcal{R}_c$ . The series for  $D(t_1, t_2; \mathcal{R}_c)$  moreover converges uniformly for all  $(t_1, t_2)$  in the plane. The Eq. (6.60) for  $w_{\text{opt}}(t; \rho_0, f_0)$  satisfies Eq. (6.54) almost everywhere<sup>††</sup> and for all  $\mathcal{R}_c$ , provided

$$D(\mathcal{R}_c) \neq 0 \quad (6.62)$$

This last condition is always satisfied in the present problem because  $D(\mu) = 0$  if and only  $(-\mu)$  is an eigenvalue<sup>†††</sup> of the kernel  $K_c(t_1, t_2)$ . In the present case, for  $\mathcal{R}_c > 0$ ,  $-\mathcal{R}_c$  can never be an eigenvalue (see the inequality 6.48) and condition (6.62) must be satisfied. If  $\mathcal{R}_c = 0$ , Eq. (6.2) is its own solution and there is no problem. The case  $\mathcal{R}_c < 0$  has no physical meaning for the present research.

The Fredholm solution contained in Eqs. (6.60) and (6.61) has a different set of attractions and difficulties,

<sup>†</sup> cf. p. 436, Korn and Korn.

<sup>††</sup> cf. pp. 176, Riesz-Nagy.

<sup>†††</sup> cf. p. 527, Margenau and Murphy.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

when compared to the Schmidt-Hilbert solution, for prospective application in the present research.

The major difference is that the Fredholm solution is given directly in terms of the known functions  $m(t)$  and  $E(\rho, f)$ , without the intermediate problem of discovering eigenfunction representations. Exploitation of this desirable feature, however, rests upon the ability to evaluate the necessary iterated kernels  $D_n(t_1, t_2)$  defined by Eq. (6.56). This may or may not be a light task, depending upon particular choices for  $m(t)$  and  $E(\rho, f)$  which define  $K_c(t_1, t_2)$ . For "realistic" functions, chosen without regard to the problems of integration, generation of the iterated kernels could be impossible in closed form.

The second potential advantage of the Fredholm solution is its validity for all  $\mathcal{R}_c$  of physical interest in the present research. If one can generate the iterated kernels, then the solution can be examined as a function of  $\mathcal{R}_c$ . However, examination of the solution in one region of great interest, namely for larger values of  $\mathcal{R}_c$ , might well require the use of many terms from the series for  $D(t_1, t_2; \mathcal{R}_c)$  in order to achieve acceptable accuracy in a finite sum. The necessity for being able to generate high-order iterated kernels is therefore doubly stressed.

### D. SOLUTION IN TERMS OF ECHO WAVEFORMS

The third, and final, general form of solution which will be developed capitalizes directly upon the particular form of the kernel  $\mathcal{K}_c(t_1, t_2)$ , Eq. (6.3). The kernel has a structure analogous to that of a kernel of finite rank, i.e., a kernel which may be written in the form

$$K(x, y) = \sum_{i=1}^r \phi_i(x) \psi_i^*(y) \quad (6.63)$$

with suitably chosen square-integrable functions  $\phi_i(x)$  and  $\psi_i(y)$ .

The following development of a general form of solution for the present case follows the line of reasoning presented by Lovitt (see Ref. 26, pp. 68-70).

The kernel  $\mathcal{K}_c(t_1, t_2)$  of Eq. (6.2) is first approximated by the following double sum:

$$\overline{\mathcal{K}}_c(t_1, t_2) = 2 \sum_{i=0}^d \sum_{j=0}^b m(t_1; \rho_i, f_j) m^*(t_2; \rho_i, f_j) \mathcal{G}(\rho_i f_j) \Delta \rho \Delta f \quad (6.64)$$

where

$$f_{i+1} = f_i + \Delta f \quad i = 0, 1, 2, \dots \quad (6.65a)$$

and

$$\rho_{j+1} = \rho_j + \Delta \rho \quad j = 0, 1, 2, \dots \quad (6.65b)$$

If now the approximate kernel  $\overline{\mathcal{K}}_c(t_1, t_2)$  is introduced into Eq. (6.2) there results, after transposition,

$$w(t_1; \rho_0, f_0) = \frac{1}{N_0} m(t_1, \rho_0, f_0) - \frac{1}{N_0} \cdot \sum_{i=0}^d \sum_{j=0}^b m(t_1; \rho_i, f_j)^2 \mathcal{G}(\rho_i, f_j) \Delta \rho \Delta f \cdot C_{ij} \quad (6.66)$$

where

$$C_{ij} = \int_{-\infty}^{\infty} m^*(t_2; \rho_i, f_j) w(t_2; \rho_0, f_0) dt_2 \quad \dagger \quad (6.67)$$

† That the desired echo parameters are here identified with  $(\rho_0, f_0)$  detracts in no way from the generality of the treatment. It is, however, a notational convenience.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

If the complex scalars  $C_{ij}$  can be determined, then Eq. (6.66) gives  $w(t_1; \rho_o, f_o)$  in the form of a weighted sum of delayed and Doppler shifted replicas of the modulation function  $n(t)$ . Apart from the  $C_{ij}$ , all other quantities on the right of Eq. (6.66) are known for any particular case of interest.

If Eq. (6.66) for  $w(t_1; \rho_o, f_o)$  is substituted into Eq. (6.67), a constraint upon the  $C_{ij}$  may be derived, namely

$$C_{kl} = \frac{1}{N_o} \Gamma_{oo}^{kl} - \frac{1}{N_o} \cdot \sum_{i=0}^d \sum_{j=0}^b \Gamma_{ij}^{kl} \cdot 2\mathcal{E}(\rho_i, f_j) \Delta\rho \Delta f \cdot C_{ij} \quad (6.68)$$

where

$$\Gamma_{ij}^{kl} = \int_{-\infty}^{\infty} m^*(t_2; \rho_k, f_l) m(t_2; \rho_i, f_j) dt_2 \quad (6.69)$$

Since Eq. (6.68) must be true for each possible pair of indices  $(k, l)$ , it implies the following set of  $(b+1)(d+1)$  linear algebraic equations for the  $C_{ij}$ :

$$\begin{bmatrix} (N_o + \Gamma_{oo}^{oo} e_{oo}) & \Gamma_{o1}^{oo} e_{o1} & \Gamma_{o2}^{oo} e_{o2} & \dots & \Gamma_{10}^{oo} e_{10} & \dots & \Gamma_{bd}^{oo} e_{bd} \\ \Gamma_{oo}^{o1} e_{oo} & (N_o + \Gamma_{o1}^{o1} e_{o1}) & \Gamma_{o2}^{o1} e_{o2} & \dots & \Gamma_{10}^{o1} e_{10} & \dots & \Gamma_{bd}^{o1} e_{bd} \\ \Gamma_{oo}^{o2} e_{oo} & \Gamma_{o1}^{o2} e_{o1} & (N_o + \Gamma_{o2}^{o2} e_{o2}) & \dots & \Gamma_{10}^{o2} e_{10} & \dots & \Gamma_{bd}^{o2} e_{bd} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Gamma_{oo}^{10} e_{oo} & \Gamma_{o1}^{10} e_{o1} & \Gamma_{o2}^{10} e_{o2} & \dots & (N_o + \Gamma_{10}^{10} e_{10}) & \dots & \Gamma_{bd}^{10} e_{bd} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Gamma_{oo}^{bd} e_{oo} & \Gamma_{o1}^{bd} e_{o1} & \Gamma_{o2}^{bd} e_{o2} & \dots & \Gamma_{10}^{bd} e_{10} & \dots & (N_o + \Gamma_{bd}^{bd} e_{bd}) \end{bmatrix} \begin{bmatrix} C_{oo} \\ C_{o1} \\ C_{o2} \\ \vdots \\ C_{10} \\ \vdots \\ C_{bd} \end{bmatrix} = \begin{bmatrix} \Gamma_{oo}^{oo} e_{oo} \\ \Gamma_{oo}^{o1} e_{oo} \\ \Gamma_{oo}^{o2} e_{oo} \\ \vdots \\ \Gamma_{oo}^{10} e_{oo} \\ \vdots \\ \Gamma_{oo}^{bd} e_{oo} \end{bmatrix} \quad (6.70)$$



where

$$e_{ij} = 2 \mathcal{E}(\rho_i, f_j) \Delta \rho \Delta f \quad (6.71)$$

A unique solution will exist for the  $C_{ij}$  if the determinant of their coefficients on the left of Eq. (6.70) is different from zero. Once the  $C_{ij}$  have been determined, Eq. (6.66) then gives a solution for the optimum weight function.

A possibly more suggestive continuous form of this solution can also be derived. To that end consider the situation as the subdivision of the  $(\rho, f)$  plane which is used in writing the approximate kernel  $\overline{K}_C(t_1, t_2)$ , Eq. (6.64), is made finer. If Eq. (6.70) continues to have a unique solution for the  $C_{ij}$  during this limiting process, and if a limit function  $C(\rho, f)$  exists for the  $C_{ij}$ , then Eq. (6.66) converges toward its continuous counterpart

$$w(t_1; \rho_0, f_0) = \frac{1}{N_0} m(t_1; \rho_0, f_0) - \frac{1}{N_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} 2 \mathcal{E}(\rho, f) C(\rho, f) m(t_1; \rho, f) d\rho df \quad (6.72)$$

and Eq. (6.68) becomes an equation for  $C(\rho, f)$ , namely

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\rho, f; r, \phi) 2 \mathcal{E}(r, \phi) C(r, \phi) dr d\phi + N_0 \cdot C(\rho, f) = G(\rho, f; \rho_0, f_0) \quad (6.73)$$

where Eq. (6.69) is replaced by

$$G(\rho, f; \rho_0, f_0) = \int_{-\infty}^{\infty} m^*(t_2; \rho, f) m(t_2; \rho_0, f_0) dt_2 \quad (6.74)$$

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Thus, by making use of the particular structure of the covariance kernel, Eq. (6.3), the problem of solving the original Eq. (6.2) is transferred to the problem of solving either the double-integral Eq. (6.73) or its discrete counterpart, Eq. (6.68). While one might envision the solution of Eq. (6.73) by application of suitable generalizations of the techniques applicable to the one-dimensional Fredholm integral equation, it is not clear that such efforts at solutions in two dimensions represent a simplification of efforts directed to solutions in one dimension. This attempt to capitalize upon the structure of the original covariance kernel seems, therefore, only to have replaced the original problem by a more difficult one in the general case. When noise is (assumed) absent, however, this method does lead to a certain insight into the nature of the optimum processor. This zero-noise solution will be discussed in Sec. B in the next chapter.

VII. ASYMPTOTIC FORMS OF SOLUTION

In the preceding chapter, several forms of solution were presented which are valid for arbitrary levels of clutter and noise interference. In this chapter the two extreme situations are considered. Solutions are given for cases where the noise interference is either much greater than the clutter, or negligible with respect to it.

A. SOLUTION WHEN NOISE INTERFERENCE IS DOMINANT

If the clutter interference is sufficiently small with respect to the noise, then the integral term on the left of Eq. (6.2) will be small with respect to  $N_0 \cdot w(t_1; \rho_0, f_0)$ . If the integral term is transposed, and the clutter-to-noise ratio  $\theta_c$  introduced, then Eq. (6.2), or (6.54), becomes

$$w(t_1; \rho_0, f_0) = \frac{1}{N_0} m(t_1; \rho_0, f_0) - \theta_c \cdot \int_{-\infty}^{\infty} K_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 \quad (7.1)$$

where  $\theta_c = \frac{2\hat{\mathcal{E}}_c}{N_0} \quad (7.2)$

$$\hat{\mathcal{E}}_c = \max_{(\rho, f)} \mathcal{E}(\rho, f) \quad (7.3)$$

$$K_c(t_1, t_2) = \frac{1}{2\hat{\mathcal{E}}_c} \cdot \mathcal{K}_c(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(\rho, f) m(t_1; \rho, f) m^*(t_2; \rho, f) d\rho df. \quad (7.4)$$

When the right-hand side of Eq. (7.1) is successively substituted for  $w(t_2; \rho_0, f_0)$  under the integral sign on the right of Eq. (7.1), the following series solution results,

$$w(t_1; \rho_0, f_0) = \frac{1}{N_0} m(t_1; \rho_0, f_0) + \frac{1}{N_0} \sum_{m=1}^{\infty} (\mathcal{K}_C)^m \int_{-\infty}^{\infty} K_C^n(t_1, t_2) m(t_2; \rho_0, f_0) dt_2 \quad (7.5)$$

$$K_C^1(t_1, t_2) \equiv K_C(t_1, t_2) \quad (7.6)$$

$$\text{and} \quad K_C^n(t_1, t_2) \equiv \int_{-\infty}^{\infty} K_C^{n-1}(t_1, \tau) K_C(\tau, t_2) d\tau. \quad (7.7)$$

It is known<sup>†</sup> that this Neumann series representation, Eq. (7.5), converges (at least in the mean) to the solution, if the kernel  $K_C(t_1, t_2)$  is Hermitian and

$$| -\mathcal{K}_C | \cdot | \lambda_{\max} | < 1 \quad (7.8)$$

where  $\lambda_{\max}$  is the greatest eigenvalue of the kernel  $K_C(t_1, t_2)$ . In the present case it has already been shown that the greatest eigenvalue of  $\mathcal{K}_C(t_1, t_2)$  does not exceed  $2\hat{\mathcal{E}}_C$  (cf. Eq. 6.48). In view of the definitions (7.3) and (7.4) it should be clear that the greatest eigenvalue of  $K_C(t_1, t_2)$ , namely  $\lambda_{\max}$ , does not exceed unity. For the present problem, therefore, the Neumann series solution is valid for  $\mathcal{K}_C$  such that

$$| \mathcal{K}_C | < \frac{1}{\lambda_{\max}} \quad (7.9)$$

where the bound  $\frac{1}{\lambda_{\max}}$  is never less than unity.

The virtue of the Neumann solution is that with Eq. (7.5) one can directly generate successively higher order approximations to  $w_{\text{opt}}(t; \rho_0, f_0)$  in terms of operations with the known

<sup>†</sup> cf. p. 435, Korn and Korn<sup>22</sup>, bearing in mind that their eigenvalues are the reciprocals of the eigenvalues defined in this research.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

functions  $K_c(t_1, t_2)$  and  $m(t; \rho_0, f_0)$ . The first two terms of Eq. (7.5) may be written, using Eq. (7.4) for  $K_c(t_1, t_2)$ , as

$$w_{opt}(t_1; \rho_0, f_0) \doteq \frac{1}{N_0} m(t_1; \rho_0, f_0) - \frac{1}{N_0} \cdot \mathcal{R}_c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(t_1; \rho, f) \hat{E}(\rho, f) g(\rho, f) d\rho df \quad (7.10)$$

$$\text{where} \quad E(\rho, f) = \hat{E}(\rho, f) / \hat{E}_c \quad (7.11)$$

$$\text{and} \quad g(\rho, f) = \int_{-\infty}^{\infty} m^*(t_2; \rho, f) m(t_2; \rho_0, f_0) dt_2. \quad (7.12)$$

The performance of this processor is given by

$$\left(\frac{S}{I}\right)_{opt} \doteq \frac{2\hat{E}_s}{N_0} \left\{ 1 - \mathcal{R}_c \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(\rho, f) |g(\rho, f)|^2 d\rho df + \dots \right\}. \quad (7.13)$$

The optimal processor is seen, in Eq. (7.10), to be written as a matched processor corrected by a term depending upon both the clutter source,  $E(\rho, f)$ , and the auto-ambiguity function,  $g(\rho, f)$ , of the modulation waveform.

The optimum performance, Eq. (7.13), is seen to fall short of the best performance in the absence of clutter, namely  $2\hat{E}_s/N_0$ , by an amount depending, again, upon the clutter source and the waveform auto-ambiguity function. Equation (7.13) confirms a goal of contemporary waveform design for maximizing  $(\frac{S}{I})$ , namely to cause  $|g(\rho, f)|$  to be small where  $\hat{E}(\rho, f)$  is large.

Note finally that Eqs. (7.10) and (7.13) are only first order approximations. They and the conclusions drawn from them are therefore only valid for relatively small  $\mathcal{R}_c$ .

1. Comparison with Matched Filter

The matched filter weight function is given by

$$w(t_1; \rho_0, f_0) = \frac{1}{N_0} m(t_1; \rho_0, f_0) \quad (7.14)$$

It is obtained directly from Eq. (6.2), for example, by setting  $\mathcal{K}_c(t_1, t_2)$  equal to zero. Using Eq. (4.33), which is valid for arbitrary interference and weight function, one may compute the signal-to-interference ratio at the matched filter output. It is given by

$$\left(\frac{S}{I}\right)_{mf} = \frac{2\hat{\mathcal{E}}_s}{N_0 + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m^*(t_1; \rho_0, f_0) \mathcal{K}_c(t_1, t_2) m(t_2; \rho_0, f_0) dt_1 dt_2} \quad (7.15)$$

or when Eq. (7.4) is introduced for  $\mathcal{K}_c(t_1, t_2)$

$$\left(\frac{S}{I}\right)_{mf} = \frac{2s}{N_0 + 2\hat{\mathcal{E}}_c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(\rho, f) |g(\rho, f)|^2 d\rho df} \quad (7.16)$$

where  $\hat{\mathcal{E}}_c$ ,  $E(\rho, f)$ , and  $g(\rho, f)$  have already been defined by Eqs. (7.3), (7.11), and (7.12), respectively.

For sufficiently small  $\hat{\mathcal{E}}_c$  one may now use Eq. (7.16) to write

$$\left(\frac{S}{I}\right)_{mf} \doteq \frac{2\mathcal{E}_s}{N_0} \left\{ 1 - \mathcal{R}_c \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(\rho, f) |g(\rho, f)|^2 d\rho df + \dots \right\} \quad (7.17)$$

Comparison of Eq. (7.10) with (7.14), and (7.13) with (7.17), leads to the conclusion that although, for small  $\mathcal{R}_c$ , the difference between matched and optimum processors is of the first order in  $\mathcal{R}_c$ , the difference in performance is only of the

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

second order in  $R_c$ . To first order in  $R_c$  the performances of matched and optimum processors are, surprisingly, the same.

The practical conclusion is that when clutter interference is small with respect to the noise, in the sense that  $R_c \ll 1$ , there is little advantage to be gained by departing from a matched processor. If there is to be a regime where marked performance differences do occur, then it must be for  $R_c > 1$ , or  $R_c \gg 1$ .

Despite these, possibly pessimistic, conclusions one should not lose sight of the fact that Eq. (7.5) presents a general solution to the problem for arbitrary functions  $\hat{E}(\rho, f)$  and  $m(t; \rho, f)$ , subject only to the constraint (7.9). This is possibly the only solution in this research which is given so simply in terms of known functions and which simultaneously has such broad generality.

## B. SOLUTION WHEN NOISE INTERFERENCE IS NEGLIGIBLE

Although it will eventually be seen that the effects of seemingly negligible noise are not necessarily negligible, first consideration will be of the case when noise is ignored and only clutter is assumed to be present.

A convenient starting point is the finite rank solution (6.66), wherein the delay and doppler variables have been quantized with increments of  $\Delta\rho$  and  $\Delta f$  respectively. When  $N_0 = 0$ , Eq. (6.70) for the  $C_{ij}$  becomes

$$\begin{bmatrix} \Gamma_{00}^{00} e_{00} & \Gamma_{01}^{00} e_{01} & \Gamma_{02}^{00} e_{02} & \cdots & \Gamma_{bd}^{00} e_{bd} \\ \Gamma_{00}^{01} e_{00} & \Gamma_{01}^{01} e_{01} & . & . & . \\ \Gamma_{00}^{02} e_{00} & \Gamma_{01}^{02} e_{01} & . & . & . \\ . & . & . & . & . \\ \Gamma_{00}^{bd} e_{00} & \Gamma_{01}^{bd} e_{01} & . & . & \Gamma_{bd}^{bd} e_{bd} \end{bmatrix} \begin{bmatrix} C_{00} \\ C_{01} \\ C_{02} \\ . \\ C_{bd} \end{bmatrix} = \begin{bmatrix} \Gamma_{00}^{00} \\ \Gamma_{00}^{01} \\ \Gamma_{00}^{02} \\ . \\ \Gamma_{00}^{bd} \end{bmatrix} \quad (7.18)$$

# COLUMBIA UNIVERSITY--ELECTRONICS RESEARCH LABORATORIES

Let it be supposed, at first, that no  $e_{ij}$  is zero. Then note that the separate  $e_{ij}$  enter the coefficient matrix of Eq. (7.18) only as multipliers of separate columns of elements. But the effect of multiplying a column of a matrix by a fixed scalar is to multiply the determinant of the matrix by that scalar. Therefore one concludes that the determinant of the coefficient matrix in Eq. (7.18) will be zero, or not, according to whether the determinant

$$\begin{vmatrix} \Gamma_{00}^{00} & \Gamma_{01}^{00} & \Gamma_{02}^{00} & \dots & \Gamma_{bd}^{00} \\ \Gamma_{00}^{01} & \Gamma_{01}^{01} & & & . \\ . & . & & & . \\ . & . & & & . \\ \Gamma_{00}^{bd} & \Gamma_{01}^{bd} & . & . & \Gamma_{bd}^{bd} \end{vmatrix} \quad (7.19)$$

is zero, or not.

This matter is finally resolved by noting that each entry in the determinant (7.19) is

$$\Gamma_{ij}^{kl} = \int_{-\infty}^{\infty} m^*(t_2; \rho_k, f_l) m(t_2; \rho_i, f_j) dt_2 \quad (7.20)$$

That is to say,  $\Gamma_{ij}^{kl}$  is the scalar product of the functions  $m(t_2; \rho_k, f_l)$  and  $m(t_2; \rho_i, f_j)$ . The determinant (7.19), containing all pair-wise scalar products for the set of functions  $\{m(t; \rho_i, f_j); i = 0, 1, \dots, d, j = 0, 1, \dots, k\}$  is, therefore, exactly Gram's determinant<sup>†</sup> for that set of functions. It is known<sup>†</sup> that Gram's determinant is zero if the set of functions

<sup>†</sup> See p. 424, Korn and Korn.<sup>22</sup>



contains linearly dependent members; it is non-zero if all the functions are linearly independent.

Retracing this argument, one can conclude that if the functions  $m(t; p_i, f_j)$  constitute a linearly independent collection, and all  $e_{ij} \neq 0$ , then the determinant of coefficients in Eq. (7.18) is non-zero and a unique solution does exist for the  $C_{ij}$ .

It may be verified<sup>†</sup> that this unique solution is

$$C_{00} = \frac{1}{e_{00}} \quad (7.21a)$$

$$\text{and } C_{ij} = 0, \quad (i,j) \neq (0,0). \quad (7.21b)$$

The optimum weight function  $w(t_1; p_0, f_0)$  cannot be determined from Eq. (6.66) in this case, because  $N_0 = 0$ , but Eqs. (7.20) and (7.21) nevertheless yield a sufficient characterization; namely

$$\int_{-\infty}^{\infty} m^*(t_2; p_0, f_0) w(t_2; p_0, f_0) dt_2 = \frac{1}{e_{00}} = \frac{1}{2\mathcal{E}(p_0, f_0) \Delta p \Delta f} \quad (7.22a)$$

$$\text{and } \int_{-\infty}^{\infty} m^*(t_2; p_i, f_j) w(t_2; p_0, f_0) dt_2 = 0, \quad (i,j) \neq (0,0). \quad (7.22b)$$

The performance for this case is also given quite simply. From Eqs. (4.38) and (7.22a) one deduces

$$\left(\frac{S}{I}\right)_{\text{opt}} = \frac{\mathcal{E}_s}{\mathcal{E}(p_0, f_0)} \cdot \frac{1}{\Delta p \Delta f}. \quad (7.23)$$

<sup>†</sup> Visual application of Cramér's rule suffices, since the right-hand member of Eq. (7.18) is proportional to the first column of coefficients.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Equations (7.22) and (7.23) provide the basis for understanding optimum detection in clutter for the ideal circumstance when  $N_0 = 0$ . They also indicate why in any realistic case noise can probably not be so completely ignored.

In the first place, note that Eq. (7.23) asserts that really the only contribution to clutter in the optimum processor output is from the clutter components originally received with delay and doppler shift identical to the delay and doppler shift (namely  $\rho_0$  and  $f_0$ ) of the echo to be detected. Discrimination against these echo-like components of clutter cannot be accomplished by a linear processor.

The implied discrimination by the optimum processor against the other clutter components i.e., those for which,  $\rho \neq \rho_0$  or  $f \neq f_0$ , however, is the exact consequence of property (7.22b) of the optimum processor. The optimum weight function is simply orthogonal to the "other" clutter components.

By virtue of the properties (7.22), the optimum processor is, apart from a scale factor, identical to one member of the set of functions "reciprocal" to the  $m(t; \rho_i, f_j)$ . To be precise, let the reciprocal set of functions be denoted by  $\{\omega(t; \rho_i, f_j); i = 0, 1, \dots, d, j = 0, 1, \dots, b\}$  and defined implicitly by

$$i) \int_{-\infty}^{\infty} m^*(t; \rho_i, f_j) \omega(t; \rho_i, f_j) dt = 1, \text{ for all } (i, j) \quad (7.24a)$$

$$ii) \int_{-\infty}^{\infty} m^*(t; \rho_i, f_j) \omega(t; \rho_k, f_l) dt = 0, \text{ for all } (k, l) \neq (i, j). \quad (7.24b)$$

Then comparison of (7.22) and (7.24) yields

$$w_{\text{opt}}(t; \rho_0, f_0) = \frac{1}{2 \mathcal{E}(\rho_0, f_0) \Delta \rho \Delta f} \cdot \omega(t; \rho_0, f_0). \quad (7.25)$$

## COLUMBIA UNIVERSITY--ELECTRONICS RESEARCH LABORATORIES

A particular example of a processor which is reciprocal to the set of signal echoes will be seen in Chapter XII. In fact, the notion that the processor must be reciprocal to the set of signal echoes (for  $N_0 = 0$ ) proves valuable in extending the results of Chapter XII to the cases considered in Chapter XIII.

The situation for  $N_0 = 0$  is thus adequately described. The suspicion that one can probably not ignore noise so completely, stems from two considerations. Equation (7.23) asserts that  $\frac{S}{I}$  must diverge to infinity if either  $\mathcal{E}(\rho_0, f_0)$  goes to zero, or if the product  $\Delta\rho\Delta f$  approaches zero.

On the first count, if the clutter output from the processor output does decrease linearly to zero with  $\mathcal{E}(\rho_0, f_0)$ , then surely a point will be reached where the predominant output interference is not clutter, but noise, and Eq. (7.23) becomes invalid.

The second cause for reservation, however, is the more interesting, as it affords some insight into the nature of an optimum processor for a low noise situation.

According to Eqs. (7.22), the optimum (reciprocal) processor has the capability of responding to a particular desired signal, while being completely insensitive to the same signal if it is displaced in delay by only so much as  $\Delta\rho$  seconds, or displaced in Doppler by only  $\Delta f$  cycles per second.

In matched filter theory one is accustomed to observing that if a delay discrimination capability of  $\Delta\rho$  seconds is to be achieved, then a system bandwidth on the order of  $(\Delta\rho)^{-1}$  cycles-per-second must be contemplated. Likewise, discrimination between signals separated by only  $\Delta f$  cycles per second in Doppler, implies signals and weight functions with durations on the order of  $(\Delta f)^{-1}$  seconds. As  $\Delta\rho\Delta f$  approaches zero, therefore, one would not be surprised if the (duration)  $\times$  (band-

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

width) product of the optimum processor became large without bound. Nor would one be surprised if as  $(\Delta\rho)^{-1}$ , and therefore processor bandwidth, became large, the optimum system was increasingly dominated by considerations of noise, no matter how small, rather than by clutter.

Cases will be seen in succeeding sections where the optimum processor achieves its performance either through large bandwidth or through long duration, or through both. The effects of even small noise levels will also be seen.

Finally, the conclusion that noise can never be neglected has far-reaching implications for the analysis. The immediate consequence is, of course, that one must solve the exact equation

$$\int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 + N_0 \cdot w(t_1; \rho_0, f_0) = m(t_1; \rho_0, f_0) \quad (7.26)$$

rather than the approximate equation

$$\int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 = m(t_1; \rho_0, f_0) . \quad (7.27)$$

Unfortunately it is not generally true that the solution to Eq. (7.26), for small  $N_0$ , closely resembles the solution to Eq. (7.27). Cases will, in fact, be seen where there is a marked divergence between the small-noise and no-noise solutions. Thus there seems to be no general way of writing the small-noise solution to Eq. (7.26) as a small perturbation of the no-noise solution to Eq. (7.27). The no-noise solution cannot, therefore, properly be called a first approximation to the true solution.

This stands in marked contrast to the situation of dominant noise described in Eqs. (7.1) through (7.13). In that situation,

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Eq. (7.10) explicitly gives the small-clutter solution as a small perturbation of the no-clutter solution (i.e., the matched processor).

### C. A HYPOTHETICAL SOLUTION

Let it finally be observed in this connection that if a signal could be designed which would be orthogonal to delayed and Doppler shifted versions of itself over some range of interest, then the reciprocal waveform space would be identical to the signal space itself. For such a signal Eq. (7.25) then indicates that the optimum processor would be exactly a matched processor, just as in the case of no clutter whatever.

It may not be surprising, therefore, that if a modulation waveform  $m(t)$  could exist which gave rise to echoes  $m(t; \rho_i, f_j)$  such that

$$\int_{-\infty}^{\infty} m^*(t; \rho_i, f_j) m(t; \rho_i, f_j) dt = 1, \quad \text{for all } (i, j) \quad (7.28a)$$

while

$$\int_{-\infty}^{\infty} m^*(t; \rho_i, f_j) m(t; \rho_k, f_l) dt = 0, \quad \text{for all } (k, l) \neq (i, j) \quad (7.28b)$$

then the solution to Eq. (6.2), with the finite rank kernel of Eq. (6.64), would be

$$w_{\text{opt}}(t; \rho_o, f_o) = \frac{1}{2\hat{\mathcal{E}}(\rho_o, f_o)\Delta\rho\Delta f + N_o} \cdot m(t; \rho_o, f_o) \quad (7.29)$$

for arbitrary levels of clutter and noise, i.e., for arbitrary  $\hat{\mathcal{E}}(\rho, f)$  and  $N_o$ . The resulting performance would be, from Eq. (4.38),

$$\left(\frac{S}{I}\right)_{\text{opt}} = \frac{2\hat{\mathcal{E}}_s}{2\hat{\mathcal{E}}(\rho_o, f_o)\Delta\rho\Delta f + N_o} \quad (7.30)$$

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

which from its form would seem to be about as well as one might expect to do with a system characterized by range resolution  $\Delta\rho$  and Doppler resolution  $\Delta f$ .

Note finally that the ratio  $\frac{S}{I}$  is insensitive to the scale factor applied to the weight function  $w(t; \rho_0, f_0)$ . Equation (7.29) can therefore equally well be written as†

$$w_{\text{opt}}(t; \rho_0, f_0) = m(t; \rho_0, f_0) \quad . \quad (7.31)$$

One therefore has the conclusion that if a "self-reciprocal" waveform existed, in the sense of Eqs. (7.28), the optimum detector would be a fixed matched filter under all circumstances of clutter and white noise. This would represent an opposite extreme to the type of solution which has, so far, seemed to arise for other types of waveforms.

---

† When the scale factor which causes  $w_{\text{opt}}(t; \rho_0, f_0)$  to be a solution of Eq. (6.2) is dropped, however, Eq. (4.38) is no longer valid for computing S/I. Equation (4.33) must be used instead.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### VIII. A MODE OF SOLUTION FOR STATIONARY CLUTTER

The solutions presented in the preceding chapter have been either general solutions of the basic integral equation, applicable for arbitrary kernels, or solutions reflecting the particular structure of the kernel of interest in this research. In this chapter a method of solution is presented which is applicable if the functions  $\mathcal{K}(\rho, f)$  and  $m(t; \rho, f)$  which define the kernel have suitable properties.

The method is a variation of the method reported by Miller and Zadeh <sup>29</sup> for solving integral equations with kernels similar to the present  $\mathcal{K}_c(t_1, t_2)$ . The variation is necessary in order that the method be more applicable to the sorts of waveforms which appear in the present radar context.

#### A. PRELIMINARY DISCUSSION

Recall that the equation to be solved is

$$\int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 + N_0 \cdot w(t_1; \rho_0, f_0) = m(t_1; \rho_0, f_0). \quad (8.1)$$

The method to be described hinges upon the ability to discover a linear operator  $P$  such that, after applying it to the functions of  $t_1$  in Eq. (8.1) to obtain

$$\int_{-\infty}^{\infty} \{P \mathcal{K}_c(t_1, t_2)\} w(t_2; \rho_0, f_0) dt_2 + N_0 \cdot \{P w(t_1; \rho_0, f_0)\} = P m(t_1; \rho_0, f_0), \quad (8.2)$$

the indicated integration may be performed to obtain

$$\int_{-\infty}^{\infty} \{P \mathcal{K}_c(t_1, t_2)\} w(t_2; \rho_0, f_0) dt_2 = Q w(t_1; \rho_0, f_0) \quad (8.3)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

where  $Q$  is some other linear operator acting with respect to  $t_1$  upon  $w(t_1; \rho_0, f_0)$ . The preceding two equations imply the following formal result

$$Qw(t_1; \rho_0, f_0) + N_0 \cdot Pw(t_1; \rho_0, f_0) = Pm(t_1; \rho_0, f_0) \quad (8.4)$$

where the integral originally appearing in Eq. (8.1) is now no longer present.

Whether the solution of Eq. (8.4) for  $w(t_1; \rho_0, f_0)$  presents an easier problem than solution of the original Eq. (8.1) depends entirely upon the operators  $P$  and  $Q$ . Whether such operators can be discovered depends, in turn, directly upon the kernel  $\mathcal{K}_c(t_1, t_2)$ .

In 1950, Zadeh and Ragazzini<sup>50</sup> applied essentially this technique to covariance kernels of the form

$$K(t_1, t_2) = K(t_1 - t_2) \quad (8.5)$$

having rational functions of frequency for their Fourier transforms (or power spectral densities). In their case, Eq. (8.4) is a linear differential equation with constant coefficients.

The extension of the method, in 1956, by Miller and Zadeh<sup>29</sup> was to more general kernels of the form

$$K(t_1, t_2) = \int_{-\infty}^{\infty} \Gamma(t_1, \lambda) \Gamma(t_2, \lambda) d\lambda \quad (8.6)$$

where  $\Gamma(t, \lambda)$  was such that linear differential operators  $L$  and  $M$ , with variable coefficients, existed with the property

$$L \Gamma(t, \lambda) = M \delta(t - \lambda) \quad (8.7)$$

In their case, Eq. (8.4) is a linear differential equation with variable coefficients.



## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

In the present case,  $L$  and  $M$  will be taken to have constant coefficients. Nevertheless, the kernel  $\mathcal{K}_c(t_1, t_2)$  remains a function of each of its arguments separately (not only of their difference), and Eq. (8.4) will be seen to be a linear difference-differential equation with, in general, variable coefficients.

### B. LINEAR DIFFERENTIAL AND DIFFERENCE OPERATORS

Let the differential operators  $L$  and  $M$ , acting upon a function  $f(t)$ , be defined by

$$Lf \equiv \left\{ a_0 + a_1 \frac{d}{dt} + \cdots + a_\ell \frac{d^\ell}{dt^\ell} \right\} f(t) \quad (8.8)$$

and

$$Mf \equiv \left\{ b_0 + b_1 \frac{d}{dt} + \cdots + b_m \frac{d^m}{dt^m} \right\} f(t) \quad (8.9)$$

where, for the present, the  $a_i$  and  $b_i$  are constant coefficients. In addition, consider the delay-superposition operator  $D$  which, for an arbitrary function  $f(t)$ , is defined by

$$Df = c_0 \cdot f(t-t_0) + c_1 \cdot f(t-t_1) + \cdots + c_d f(t-t_d) \quad (8.10)$$

These operators will be observed to all yield functions of  $t$  as results of their application. Furthermore, it is readily shown that they commute, for just as

$$a_\ell \frac{d^\ell}{dt^\ell} \left\{ b_m \frac{d^m}{dt^m} f(t) \right\} \equiv b_m \frac{d^m}{dt^m} \left\{ a_\ell \frac{d^\ell}{dt^\ell} f(t) \right\} \quad (8.11)$$

so also

$$a_\ell \frac{d^\ell}{dt^\ell} \left\{ c_d f(t-t_d) \right\} \equiv c_d \cdot \left\{ a_\ell \frac{d^\ell}{dt^\ell} f(t) \right\} \Big|_{t-t_d} \quad (8.12)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

These equalities lead to the respective conclusions that

$$LMf \equiv MLf \quad (8.13)$$

and

$$LDf \equiv DLf \quad (8.14)$$

One may also verify, by repeated integration by parts, and the repeated assumption that

$$\left[ \delta^{(v-1)}(t-t_0) f^{(m-v+1)}(t) \right]_{t=-\infty}^{t=+\infty} = 0 \quad v = 1, 2, \dots, n, \quad (8.15)$$

$$\text{that } \int_{-\infty}^{+\infty} \delta^{(m)}(t-t_0) f(t) dt = (-1)^n f^{(n)}(t_0) \quad (8.16)$$

$$\text{where } \delta^{(v)}(t) = \frac{d^v}{dt^v} \delta(t) \quad (8.17)$$

$$\text{and } f^{(n)}(t) = \frac{d^n}{dt^n} f(t) \quad (8.18)$$

The change of variable  $\tau = t_0 - t$  in Eq. (8.17) leads ultimately to the complementary conclusion

$$\int_{-\infty}^{\infty} \delta^{(n)}(t_0-t) f(t) dt = f^{(n)}(t_0) \quad (8.19)$$

From Eqs. (8.8), (8.16), and (8.19) one can deduce that, for the differential operators,

$$\int_{-\infty}^{\infty} f(t) \left\{ L_t \delta(t-\tau) \right\} dt \equiv \tilde{L}_\tau f(\tau) \quad (8.20a)$$

$$\text{and } \int_{-\infty}^{\infty} f(t) \left\{ L_t \delta(\tau-t) \right\} dt \equiv L_\tau f(\tau) \quad (8.20b)$$

$$\text{where } \tilde{L}f \equiv \left\{ a_0 - a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} - \dots + (-1)^l \frac{d^l}{dt^l} \right\} f(t). \quad (8.20c)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

These results will be used presently for their clear utility in removing indicated integrations. Also to be used is the fact that, in terms of  $\tilde{L}$  and  $\tilde{M}$ , the respective adjoint operators to  $L$  and  $M$ , the adjoint of the composition of  $L$  and  $M$  is given by

$$(\widetilde{LM}) f \equiv \tilde{M} \tilde{L} f \quad (8.21)$$

The reverse order on the right is used here, although not necessary, in order to facilitate later comparison with the results of Miller and Zadeh<sup>29</sup>, where  $L$  and  $M$  have variable coefficients and the reverse order is the only correct order.

The symbolic inverse  $L^{-1}$  to the operator  $L$  may be defined by

$$L^{-1} f = \int_{-\infty}^{\infty} L^{-1}(t_1, t_2) f(t_2) dt_2 \quad (8.22a)$$

where  $L^{-1}(t_1, t_2)$  satisfies the differential equation

$$L_{t_1} L^{-1}(t_1, t_2) = \delta(t_1 - t_2) \quad (8.22b)$$

The consistency of the definition is verified by using these equations with Eq. (8.16) to deduce

$$L_{t_1} L_{t_1}^{-1} f(t_1) = \int_{-\infty}^{\infty} L_{t_1} L^{-1}(t_1, t_2) f(t_2) dt_2 = \int_{-\infty}^{\infty} \delta(t_1 - t_2) f(t_2) dt_2 = f(t_1). \quad (8.22c)$$

A useful operator  $D^+$ , related to the operator  $D$  defined by Eq. (8.10), is given by

$$D^+ f = c_0 f(t+t_0) + c_1 \cdot f(t+t_1) + \cdots + c_d \cdot f(t+t_d) \quad (8.23a)$$

The particular property of interest for the sequel is

$$D_{t_2} f(t_2 - t_1) \equiv D_{t_1}^+ f(t_2 - t_1) \quad (8.23b)$$

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Finally, let it be noted that complex conjugation, indicated by an asterisk, has the following notational consequences. For the differential operators

$$(Lf)^* = L^*f^* = \left\{ a_0^* + a_1^* \frac{d}{dt} + \cdots + a_l^* \frac{d^l}{dt^l} \right\} f^*(t) \quad (8.24a)$$

while for the delay operator

$$(Df)^* = D^*f^* = c_0^* \cdot f^*(t-t_0) + c_1^* \cdot f^*(t-t_1) + \cdots + c_d^* \cdot f^*(t-t_d). \quad (8.24b)$$

The meanings to be attached to  $L^*f$  and  $D^*f$  are readily determined by consideration of  $(Lf^*)^*$  and  $(Df^*)^*$  respectively. Thus

$$L^*f = \left\{ a_0^* + a_1^* \frac{d}{dt} + \cdots + a_l^* \frac{d^l}{dt^l} \right\} f(t) \quad (8.25a)$$

and

$$D^*f = c_0^* \cdot f(t-t_0) + c_1^* \cdot f(t-t_1) + \cdots + c_d^* \cdot f(t-t_d) \quad (8.25b)$$

With the necessary basic relations for the operators now having been described, attention may once again be directed to the problem of solving Eq. (8.1).

### C. THE EQUIVALENT DIFFERENCE-DIFFERENTIAL EQUATION

Let it be supposed that the clutter source is essentially stationary and that the dispersion function  $\mathcal{E}(\rho, f)$  may therefore be represented in the form

$$\mathcal{E}(\rho, f) = \hat{\mathcal{E}}_c \cdot E_c(\rho) \cdot \delta(f) \quad (8.26)$$

where, as before,  $\hat{\mathcal{E}}_c = \max_{\rho} \mathcal{E}_c(\rho) \quad (8.27)$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

For this dispersion function, the clutter covariance function

$\mathcal{K}_c(t_1, t_2)$  is given by

$$\mathcal{K}_c(t_1, t_2) = 2\hat{\mathcal{E}}_c \int_{-\infty}^{\infty} E_c(\rho) m(t_1; \rho) m^*(t_2; \rho) d\rho \quad (8.28)$$

where Eqs. (6.3) and (8.16) have been used, and notation has been shortened by defining

$$m(t; \rho) \equiv m(t; \rho, 0) \equiv m(t - \rho) \quad (8.29)$$

Let it further be supposed that for the modulation envelope  $m(t)$ , linear operators  $L, M$ , and  $D$  with constant coefficients exist such that

$$L_t m(t) = M_t D_t \delta(t) \quad (8.30)$$

The subscripts attached to the operators indicate explicitly the independent argument being acted upon.

Using Eqs. (8.28) and (8.30) one may therefore write

$$L_{t_1} \mathcal{K}_c(t_1, t_2) = 2\hat{\mathcal{E}}_c \int_{-\infty}^{\infty} \left\{ L_{t_1} \delta(t_1 - \rho) \right\} E_c(\rho) m^*(t_2; \rho) d\rho \quad (8.31)$$

$$= 2\hat{\mathcal{E}}_c \int_{-\infty}^{\infty} \left\{ M_{t_1} D_{t_1} \delta(t_1 - \rho) \right\} E_c(\rho) m^*(t_2; \rho) d\rho \quad (8.32)$$

where the fact that  $L, M$ , and  $D$  have constant coefficients enables one to write from Eq. (8.30),

$$L_t m(t - \rho) \equiv M_t D_t \delta(t - \rho) \quad (8.33)$$

The integration indicated in Eq. (8.32) may now be performed to obtain, with the aid of Eq. (8.19),

$$L_{t_1} \mathcal{K}_c(t_1, t_2) = 2\hat{\mathcal{E}}_c M_{t_1} D_{t_1} \left\{ E_c(t_1) m^*(t_2; t_1) \right\} \quad (8.34)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Having now come to this partial result, one can use it to write

$$L_{t_1} \int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 \equiv 2\hat{\mathcal{G}}_c \cdot M_{t_1} D_{t_1} E_c(t_1) \int_{-\infty}^{\infty} m^*(t_2; t_1) w(t_2; \rho_0, f_0) dt_2. \quad (8.35)$$

The integration on the right, with respect to  $t_2$ , will be removed by writing symbolically, from Eqs. (8.29), (8.33) and (8.23)

$$\begin{aligned} m(t_2; t_1) &= L_{t_2}^{-1} M_{t_2} D_{t_2} \delta(t_2 - t_1) \\ &= L_{t_2}^{-1} M_{t_2} D_{t_1}^+ \delta(t_2 - t_1). \end{aligned} \quad (8.36)$$

Using Eqs. (8.20) and (8.36) one may now write, for the right hand side of Eq. (8.35)

$$\int_{-\infty}^{\infty} \left\{ L_{t_2}^{-1} M_{t_2} D_{t_2} \delta(t_2 - t_1) \right\} w(t_2; \rho_0, f_0) dt_2 = \tilde{M}_{t_1}^* \tilde{L}_{t_1}^{-1} D_{t_1}^+ w(t_1; \rho_0, f_0). \quad (8.37)$$

This result, together with Eq. (8.35), yields

$$L_{t_1} \int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) w(t_2; \rho_0, f_0) dt_2 \equiv 2\hat{\mathcal{G}}_c \cdot M_{t_1} D_{t_1} E_c(t_1) \tilde{M}_{t_1}^* \tilde{L}_{t_1}^{-1} D_{t_1}^+ w(t_1; \rho_0, f_0) \quad (8.38)$$

to serve as a replacement for the first term of Eq. (8.1) after  $L_{t_1}$  has been applied to it.

Thus, by applying  $L_{t_1}$  to each side of Eq. (8.1) and using Eq. (8.38) for the first term, one arrives at

$$2\hat{\mathcal{G}}_c M_{t_1} D_{t_1} E_c(t_1) \tilde{M}_{t_1}^* \tilde{L}_{t_1}^{-1} D_{t_1}^+ w(t_1; \rho_0, f_0) + N_0 \cdot L_{t_1} w(t_1; \rho_0, f_0) = L_{t_1} m(t_1; \rho_0, f_0) \quad (8.39)$$

as the equation to be solved for  $w(t; \rho_0, f_0)$ .

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## D. DISCUSSION OF THE SOLUTION

The first thing to consider with respect to Eq. (8.39) is the set of circumstances for which it reduces to an equation consistent with the work of Miller and Zadeh.<sup>29</sup> To this end, suppose that

$$i) \quad 2\hat{\mathcal{E}}_c \cdot E_c(t) \equiv 1, \quad (8.40a)$$

$$ii) \quad D_t f(t) \equiv f(t), \quad (8.40b)$$

$$iii) \quad N_0 = 0. \quad (8.40c)$$

Then Eq. (8.39) becomes

$$M_t M_t^* L_t^{-1} w(t; \rho_0, f_0) = L_t m(t; \rho_0, f_0). \quad (8.41)$$

If, on the other hand, one uses their Eq. (5), which for the present case becomes

$$w(t; \rho_0, f_0) = \int_{-\infty}^{\infty} \mathcal{K}_c^{-1}(t, \xi) m(\xi; \rho_0, f_0) d\xi \quad (8.42)$$

together with their Eq. (24), which is

$$M_t \tilde{M}_t \tilde{L}_t^{-1} \mathcal{K}_c^{-1}(t, \xi) = L_t \delta(t - \xi) \quad (8.43)$$

in the present case, then upon multiplying both sides of Eq. (8.43) by  $m(\xi; \rho_0, f_0)$  and integrating with respect to  $\xi$ , one again arrives at (essentially) Eq. (8.41) above.<sup>†</sup> In the process of transcribing Eqs. (8.42) and (8.43), those terms which arose in Miller and Zadeh<sup>29</sup> because the solution  $w(t; \rho_0, f_0)$

<sup>†</sup> Miller and Zadeh<sup>29</sup> analyze an equation with real waveforms. The complex notation of Eq. (8.41) is, therefore, not reproduced by the process described.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

was to be restricted to a finite interval  $(a,b)$ , are neglected for the present case of the doubly infinite interval  $(-\infty, \infty)$ .

Thus, by restricting the present analysis in one fashion and restricting the analysis of Miller and Zadeh<sup>29</sup> in another fashion, both analyses may be shown to include as a special case, the common simpler problem defined by Eq. (8.41).

The solution of Eq. (8.41) can be written formally in terms of the individual operators  $L$  and  $M$ . This cannot be done for the more general Eq. (8.39), however, because of the presence of the term proportional to  $N_0$ . The problem of solving Eq. (8.39) can however be described in somewhat more precise, though not explicit, terms. The discussion parallels that given by Miller and Zadeh.<sup>29</sup>

Let there be introduced an auxiliary function  $\psi(t; \rho_0, \dot{z}_0)$  defined symbolically by

$$\psi(t; \rho_0, f_0) = \tilde{L}_t^{-1*} w(t; \rho_0, f_0), \quad (8.44)$$

and from which  $w(t; \rho_0, f_0)$  can be recovered by the inverse relation

$$w(t; \rho_0, f_0) = \tilde{L}^* \psi(t; \rho_0, f_0). \quad (8.45)$$

In terms of this auxiliary function, Eq. (8.39) can be re-expressed as

$$\left\{ 2\hat{E}_c M_t D_t E_c(t) \tilde{M}_t^* D_t^{+*} + N_0 \cdot L_t \tilde{L}_t^* \right\} \psi(t; \rho_0, f_0) = L_t m(t; \rho_0, f_0) \quad (8.46)$$

where the commutativity of  $D_t$  and  $L_t$  has been used (cf. Eq. 8.14).

The operator appearing in braces in the preceding equation is seen to be a difference-differential operator with variable coefficients supplied by the function  $E_c(t)$ . If that equation



can be solved for  $\psi(t; \rho_o, f_o)$ , then the solution  $w(t; \rho_o, f_o)$  for the original problem is directly available from Eq. (8.45). Indeed, if  $\psi_o(t; \rho_o, f_o)$  represents the "particular solution" to Eq. (8.46) and  $\psi_j(t; \rho_o, f_o)$  are linearly independent solutions to the homogeneous equation

$$\left\{ 2\hat{E}_c^* M_t D_t E_c(t) \tilde{M}_t^* D_t + N_o \cdot L_t \tilde{L}_t^* \right\} \psi(t; \rho_o, f_o) = 0 \quad (8.47)$$

then the optimal processor weight function is given symbolically by

$$w_{opt}(t; \rho_o, f_o) = \tilde{L}^* \psi_o(t; \rho_o, f_o) + \sum_j C_j \cdot \tilde{L}^* \psi_j(t; \rho_o, f_o) \quad (8.48)$$

The undetermined coefficients  $C_j$  are to be chosen such that when  $w_{opt}(t; \rho_o, f_o)$  given by equation (8.48) is substituted into the original equation (8.1), the resulting expression is an identity. This is a procedure also followed in solving problems of stationary interference.<sup>†</sup>

It is seen, therefore, that the problem of solving the original integral equation (8.1) has by this procedure been converted to the problem of solving a linear difference-differential equation with variable coefficients, namely equation (8.39). Chapters XII and XIII contain examples of this mode of solution.

Note particularly that the equation which must be solved in an individual case depends entirely upon the underlying

---

<sup>†</sup> cf. Zadeh and Ragazzini<sup>50</sup>, or Helstrom,<sup>15</sup> pp. 109-114.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

modulation function  $m(t)$  through the linear operators  $L_t$ ,  $M_t$ , and  $D_t$ . A change of  $m(t)$ , therefore, changes not only the right side of equation (8.46), but also changes at least the coefficients, and possibly also the terms, which appear on the left side.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### IX. A GENERAL PERFORMANCE IMPROVEMENT BOUND

This chapter presents an upper bound to the amount by which the performance of an optimum processor may exceed the performance of the matched processor for the same situation. This bound results from the fact that in Chapter VI upper and lower bounds could be derived for the eigenvalues of the clutter covariance kernel.

#### A. DERIVATION

It has already been shown in Chapter VI that the optimum performance for a linear processor may be written in the form

$$\left(\frac{S}{I}\right)_{\text{opt}} = 2\mathcal{E}_s \cdot \sum_{k=0}^{\infty} \frac{|\alpha_k|^2}{\mu_k + N_o} \quad (9.1)$$

where the liberty has been taken of introducing the new quantities

$$|\alpha_o|^2 \equiv (m_o, m_o) \quad (9.2a)$$

and

$$\mu_o = 0 \quad (9.2b)$$

If these definitions are used in Eq. (9.1) and the first term ( $k = 0$ ) of the sum indicated separately, then it is clear that Eqs. (9.1) and (6.31) are identical.

Matched filter performance can also be derived in terms of eigenvalues  $\mu_k$  and the coefficients  $\alpha_k$ , after a few preliminary calculations. The matched filter is defined by

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$w_{mf}(t; \rho_o, f_o) = \frac{1}{N_o} m(t; \rho_o, f_o) \quad (9.3)$$

where, it will be recalled,

$$\int_{-\infty}^{\infty} |m(t; \rho_o, f_o)|^2 dt = 1 \quad (9.4)$$

These two equations therefore imply that, for the numerator of Eq. (4.33),

$$\left| \int_{-\infty}^{\infty} w_{mf}^*(t; \rho_o, f_o) m(t; \rho_o, f_o) dt \right|^2 = \frac{1}{N_o^2} \quad (9.5)$$

Using next Eqs. (6.23), (6.17), and (6.18), one finds that

$$\int_{-\infty}^{\infty} \mathcal{K}_c(t_1, t_2) w_{mf}(t_2; \rho_o, f_o) dt_2 = \frac{1}{N_o} \cdot \sum_{k=1}^{\infty} \alpha_k \cdot u_k \cdot \phi_k(t_1) \quad (9.6)$$

which in turn leads to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_{mf}^*(t_1; \rho_o, f_o) \mathcal{K}_c(t_1, t_2) w_{mf}(t_2; \rho_o, f_o) dt_1 dt_2 = \frac{1}{N_o^2} \sum_{k=1}^{\infty} u_k |\alpha_k|^2 \quad (9.7)$$

Finally,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_{mf}^*(t_1; \rho_o, f_o) \delta(t_1 - t_2) w_{mf}(t_2; \rho_o, f_o) dt_1 dt_2 = \frac{1}{N_o^2} \quad (9.8)$$

Since the covariance kernel for the total interference  $\mathcal{K}(t_1, t_2)$  is given by

$$\mathcal{K}(t_1, t_2) = \mathcal{K}_c(t_1, t_2) + N_o \cdot \delta(t_1 - t_2) \quad (9.9)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

one may use Eqs. (9.5) and (9.7) through (9.9), together with Eq. (4.33) to find

$$\left(\frac{S}{I}\right)_{mf} = \frac{2\mathcal{E}_s}{N_0 + \sum_{k=1}^{\infty} \mu_k \cdot |a_k|^2} \quad (9.10)$$

However, since

$$\int_{-\infty}^{\infty} |m(t; \rho_0, f_0)|^2 dt \equiv \sum_{k=0}^{\infty} |a_k|^2 = 1, \quad (9.11)$$

one may write

$$\left(\frac{S}{I}\right)_{mf} = \frac{2\mathcal{E}_s}{N_0 \sum_{k=0}^{\infty} |a_k|^2 + \sum_{k=1}^{\infty} \mu_k \cdot |a_k|^2} \quad (9.12)$$

This may finally be written, using Eq. (9.2b), as

$$\left(\frac{S}{I}\right)_{mf} = \frac{2\mathcal{E}_s}{\sum_{k=0}^{\infty} (N_0 + \mu_k) \cdot |a_k|^2} \quad (9.13)$$

Equations (9.1) and (9.13) now yield the intermediate result

$$\frac{\left(\frac{S}{I}\right)_{opt}}{\left(\frac{S}{I}\right)_{mf}} = \left\{ \sum_{k=0}^{\infty} \frac{|a_k|^2}{(N_0 + \mu_k)} \right\} \cdot \left\{ \sum_{k=0}^{\infty} (N_0 + \mu_k) \cdot |a_k|^2 \right\}, \quad (9.14)$$

to which Kantorovich's inequality may be applied directly. Kantorovich showed† that

† See either p. 142 of Kantorovich's article,<sup>19</sup> or Appendix C of this dissertation.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$\left\{ \sum_k \gamma_k^2 \right\} \cdot \left\{ \sum_k \gamma_k^{-1} \mu_k^2 \right\} \leq \frac{1}{4} \left[ \left( \frac{M}{m} \right)^{\frac{1}{2}} + \left( \frac{m}{M} \right)^{\frac{1}{2}} \right]^2 \left( \sum_k \mu_k^2 \right)^2 \quad (9.15)$$

provided  $0 < m \leq \gamma_k \leq M$  . (9.16)

The bounds for the eigenvalues  $\mu_k$ , given by the inequalities (6.48), are now used to establish the necessary bounds for the  $(N_o + \mu_k)$ , namely

$$0 < N_o \leq N_o + \mu_k \leq N_o + 2\hat{\mathcal{E}}_c, \quad k = 0, 1, 2, \dots \quad (9.17)$$

Therefore, Eq. (9.14) used with the inequality (9.14) yields

$$\frac{\left( \frac{S}{I} \right)_{\text{opt}}}{\left( \frac{S}{I} \right)_{\text{mf}}} \leq \frac{1}{4} \left[ \left( \frac{N_o + 2\hat{\mathcal{E}}_c}{N_o} \right)^{\frac{1}{2}} + \left( \frac{N_o}{N_o + 2\hat{\mathcal{E}}_c} \right)^{\frac{1}{2}} \right]^2, \quad (9.18)$$

which, except for rearrangement, is the final result. If one introduces the clutter-to-noise ratio parameter  $\mathcal{R}_c$  defined by

$$\mathcal{R}_c = \frac{2\hat{\mathcal{E}}_c}{N_o} \quad (9.19)$$

then the bound on the right of the inequality (9.18) is seen to be a function only of  $\mathcal{R}_c$ . Thus one may write the inequality in the form

$$\left( \frac{S}{I} \right)_{\text{opt}} \leq \left( \frac{S}{I} \right)_{\text{mf}} \cdot B(\mathcal{R}_c) \quad (9.20)$$

where

$$B(\mathcal{R}_c) = \frac{1}{4} \left[ (1 + \mathcal{R}_c)^{\frac{1}{2}} + (1 + \mathcal{R}_c)^{-\frac{1}{2}} \right]^2 \quad (9.21)$$

## B. DISCUSSION

From the inequality (9.20) one sees that the extent to which signal-to-interference ratio may be improved by departing from a matched processor is strictly limited by the bound  $B(\mathcal{R}_c)$  given by Eq. (9.21).

As the notation indicates, the bound depends only upon the clutter-to-noise-ratio parameter  $\mathcal{R}_c$ . Figure 4 shows the variation of the bound as a function of  $\mathcal{R}_c$ . By appropriately expanding the right side of Eq. (9.1) one may verify the asymptotic behavior shown in Fig. 4; namely

$$\text{i) for } \mathcal{R}_c \rightarrow 0, \quad B(\mathcal{R}_c) \rightarrow 1 + \frac{1}{4} \mathcal{R}_c^2, \quad (9.22)$$

while

$$\text{ii) for } \mathcal{R}_c \rightarrow \infty, \quad B(\mathcal{R}_c) \rightarrow 1 + \frac{1}{4} \mathcal{R}_c. \quad (9.23)$$

The asymptotic behavior of  $B(\mathcal{R}_c)$  for small  $\mathcal{R}_c$ , given by Eq. (9.22), supports the conclusion deduced earlier from Eqs. (7.13) and (7.17). For small clutter-to-noise ratios the performances of optimum and matched processors can differ only by an amount proportional to the square of  $\mathcal{R}_c$ . As shown explicitly by Eqs. (7.13) and (7.17), and implied by Eq. (9.22),  $(\frac{S}{I})_{\text{opt}}$  and  $(\frac{S}{I})_{\text{mf}}$  as functions of  $\mathcal{R}_c$  must be identical up to and including terms proportional to  $\mathcal{R}_c$ , for  $\mathcal{R}_c$  near zero.

For  $\mathcal{R}_c$  sufficiently large, however, one can write

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

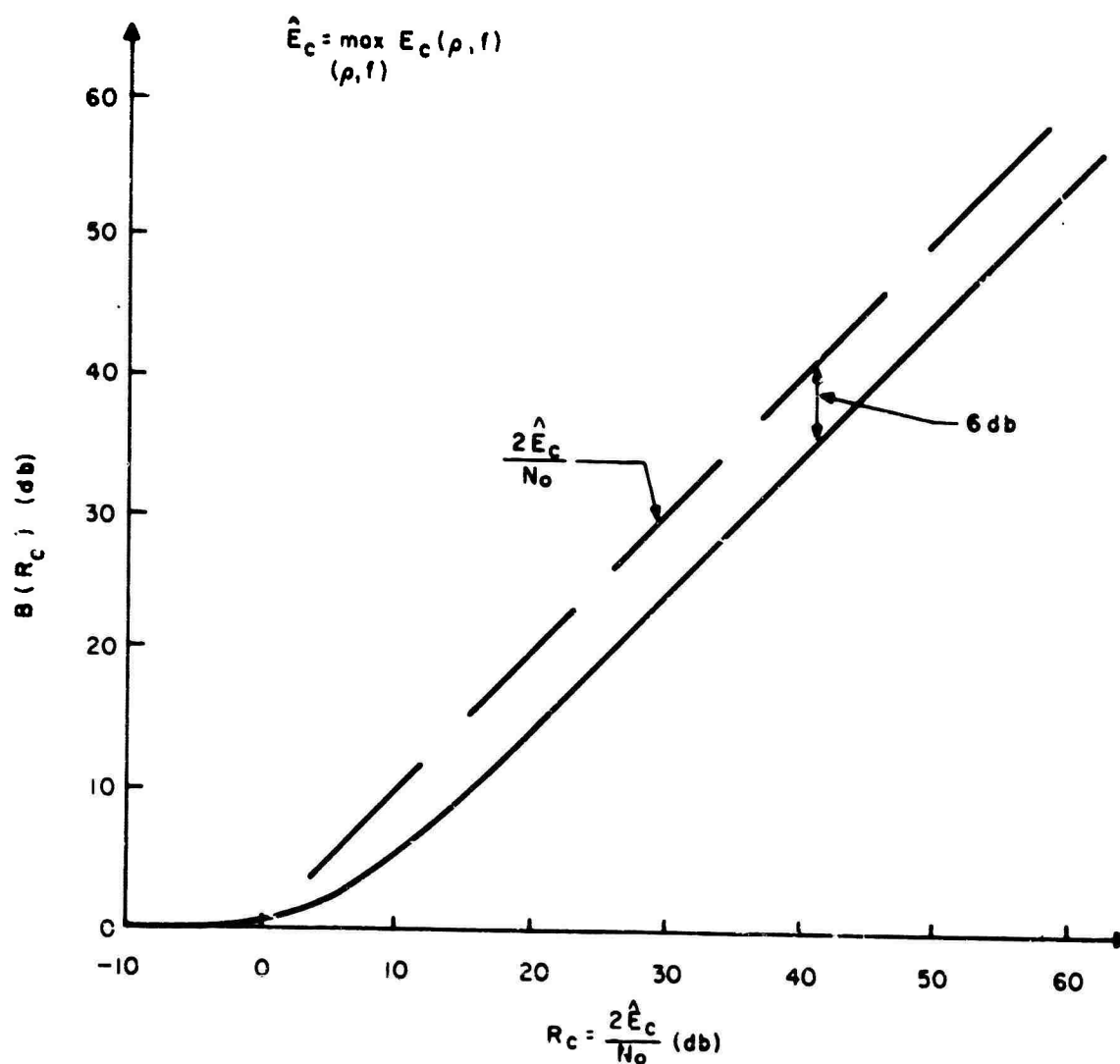


FIG. 4 VARIATION OF PERFORMANCE IMPROVEMENT BOUND WITH CLUTTER-TO-NOISE-RATIO PARAMETER



COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$10 \log \left( \frac{S}{I} \right)_{\text{opt}} - 10 \log \left( \frac{S}{I} \right)_{\text{mf}} \leq 10 \log R_c - 10 \log 4 \quad (9.24a)$$

or

$$\left( \frac{S}{I} \right)_{\text{opt}} (\text{db}) - \left( \frac{S}{I} \right)_{\text{mf}} (\text{db}) \leq R_c (\text{db}) - 6 \text{ db} \quad (9.24b)$$

where Eq. (9.23) and the inequality (9.20) have been combined. For large clutter-to-noise ratios, therefore, the potential improvement open to an optimum processor is, except for 6 decibels, exactly the clutter-to-noise ratio parameter  $R_c$ .

Whether the maximum performance improvement can be achieved or not is an entirely different question. Two conditions can be shown to be necessary.† Strict equality will be achieved in Eq. (9.20) only if

- i) the echo to be detected,  $m(t; \rho_o, f_o)$ , is exactly given by

$$m(t; \rho_o, f_o) = \frac{1}{\pm \sqrt{2}} \phi_k(t) + \frac{1}{\pm \sqrt{2}} \phi_\ell(t) ; \quad (9.25)$$

- and ii) the least and greatest eigenvalues are  $\mu_k$  and  $\mu_\ell$  respectively, with values

$$\mu_k = 0 \quad (9.26a)$$

and

$$\mu_\ell = 2 \hat{\mathcal{E}}_c . \quad (9.26b)$$

It is to be expected that these conditions will only rarely, if ever, be achieved in a practical situation. In the first place, there is little to guarantee that the upper and lower limits of Eqs. (9.26) will indeed correspond to eigenvalues of the clutter covariance kernel  $\mathcal{K}_c(t_1, t_2)$ . In

† See Appendix C.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

the second place, even if Eq. (9.25) were satisfied for some particular parameter pair  $(\rho_o, f_o)$ , there is little reason to expect a similar representation of  $m(t; \rho, f)$  for another pair of echo parameters. These considerations stem in large part from the generality of the bound.

That the bound is generally applicable over the entire scope of this research problem, should be noted. It is applicable for arbitrary modulation functions  $m(t)$ , arbitrary clutter dispersion functions  $\mathcal{E}(\rho, f)$ , and arbitrary white-noise levels  $N_o$ . It is indeed independent of any detailed characteristics of the modulation and dispersion functions. These virtues have as their consequence the likelihood that the bound will be conservative (i.e., too great) in individual cases.

Finally, from the fact that the bound is a function only of the parameter  $\mathcal{R}_c$ , one may once again deduce the importance of noise in limiting optimum system performance. It is only, but indeed, the presence of noise which causes both  $\mathcal{R}_c$  and the bound itself to be finite. In consequence, no matter how great the clutter, system performance will be limited by  $B(\mathcal{R}_c)$ .

X. GAUSSIAN ECHOES IN UNIFORMLY  
EXTENDED GAUSSIAN CLUTTER

This chapter describes the optimal processor for detecting a doppler shifted radar echo, with an envelope the shape of the Gaussian function, in uniformly extended clutter which has a Gaussian frequency dispersion function. The signal-to-clutter ratio, optimal processor frequency response function, and the optimal cross-ambiguity function are found and discussed.

These results are compared with the results already derived by Westerfield, et al,<sup>46</sup> for a matched filter receiver designed for the identical detection problem.

Because the functions which characterize this problem have been assumed Gaussian, the results for both optimal and matched processors can be found analytically and in closed form when noise is neglected.

A. ANALYTIC RESULTS

The pulse modulating function  $m(t)$  is assumed to be given by

$$m(t) = 2^{\frac{1}{4}} W^{\frac{1}{2}} \cdot \exp \{ -\pi W^2 t^2 \}, \quad (10.1)$$

where the amplitude has been chosen to satisfy the energy constraint

$$\int_{-\infty}^{\infty} m^*(t)m(t)dt = 1 \quad (10.2)$$

for all values of the bandwidth parameter  $W$ . The Fourier transform of  $m(t)$  is given by

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$M(f) = \frac{2^{\frac{1}{4}}}{W^{\frac{1}{2}}} \cdot \exp \left\{ -\pi \frac{f^2}{W^2} \right\} . \quad (10.3)$$

From this one concludes that the power spectrum of the modulation is given by

$$| M(f) |^2 = \frac{2^{\frac{1}{2}}}{W} \cdot \exp \left\{ -\pi \cdot 2 \frac{f^2}{W^2} \right\} \quad (10.4)$$

and that required autocorrelation function  $m(\tau)$  is

$$m(\tau) = \mathcal{F}^{-1} \{ | M(f) |^2 \} \quad (10.5)$$

or

$$m(\tau) = \exp \left\{ -\pi \tau^2 \cdot \frac{W^2}{2} \right\} . \quad (10.6)$$

The dispersion function for the clutter source is given by

$$\mathcal{E}_c(\rho, f) = \mathcal{E}_c \cdot Q(f) \quad (10.7)$$

where

$$Q(f) = \frac{1}{W_q} \exp \left\{ -\pi \frac{f^2}{W_q^2} \right\} . \quad (10.8)$$

The received clutter is therefore assumed to arise from an extensive distribution of scattering centers which have a Gaussian distribution of radial velocities.

It may be verified for this clutter model that

$$\int_{-\infty}^{\infty} Q(f) df = 1 , \quad (10.9)$$

and that the associated correlation function  $Q(\tau)$ , given in general by

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$Q(\tau) = \mathcal{F}^{-1} \{Q(f)\} , \quad (10.10)$$

is

$$Q(\tau) = \exp \{ -\pi \tau^2 w_q^2 \} . \quad (10.11)$$

The correlation function  $\mathcal{K}_c(\tau)$  for the received clutter process is found using Eqs. (4.45), (10.6), and (10.11). The result is

$$\mathcal{K}_c(\tau) = 2 \mathcal{E}_c \cdot \exp \{ -\pi \tau^2 w_k^2 \} \quad (10.12)$$

where 
$$w_k^2 = w_q^2 + \frac{1}{2} w^2 . \quad (10.13)$$

The received clutter power spectral density, defined at Eq. (4.56) as

$$K_c(f) = \mathcal{F} \{ \mathcal{K}_c(\tau) \} , \quad (10.14)$$

is given by

$$K_c(f) = 2 \mathcal{E}_c \cdot \frac{1}{w_k} \exp \left\{ -\pi \frac{f^2}{w_k^2} \right\} . \quad (10.15)$$

It may be observed in passing that, since  $K_c(f)$  is the Fourier transform of the product of  $m(\tau)$  and  $Q(\tau)$ ,  $K_c(f)$  might alternatively be computed as the convolution of  $|M(f)|^2$  and  $Q(f)$ . In the present case the calculation outlined above seems simpler. The general fact remains, however, that the clutter power spectrum is the convolution of the signal power spectrum with the clutter dispersion function in uniformly extended clutter.

The final result needed for computation of system performance is an expression for the Fourier transform of the echo to be detected. In the present case, for  $M(f)$  given by (10.3), the result is

$$M(f; \rho_o, f_o) = \frac{2^{\frac{1}{4}}}{w^{\frac{1}{2}}} \cdot \exp \left\{ -\pi \frac{(f-f_o)^2}{w^2} \right\} \cdot \exp \{ -j2\pi f \rho_o \} . \quad (10.16)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

One can now compute the output signal-to-interference ratio for this case. When noise is ignored, Eq. (4.51) becomes

$$\frac{S}{I} = 2\mathcal{E}_s \cdot \int_{-\infty}^{\infty} \frac{|M(f; \rho_o f_o)|^2}{K_c(f)} df \quad (10.17)$$

Together with Eqs. (10.15) and (10.16) it yields

$$\left(\frac{S}{I}\right)_{\text{opt}} = \frac{\mathcal{E}_s}{\mathcal{E}_c} W_q \left(1 + \frac{W^2}{2W_q^2}\right) \cdot \exp\left\{\pi \frac{f_o^2}{W_q^2}\right\} \quad (10.18)$$

This result may be compared directly to the corresponding result for a matched filter processor which was given by Westerfield, et al.<sup>4e</sup> Their result, in an altered form, is

$$\left(\frac{S}{I}\right)_{\text{mf}} = \frac{\mathcal{E}_s}{\mathcal{E}_c} W_q \left(1 + \frac{W^2}{W_q^2}\right)^{\frac{1}{2}} \cdot \exp\left\{\pi \frac{f_o^2}{W_q^2 + W^2}\right\} \quad (10.19)$$

The discussion and comparison of these and subsequent results in this section will be deferred to the next section of this chapter.

The frequency response function for the optimal processor is shown in Appendix B to be given in general by

$$H_{\text{opt}}(f; \rho_o, f_o) = W^*(f; \rho_o, f_o) = \frac{M^*(f; \rho_o, f_o)}{K(f)} \quad (10.20)$$

where

$$\begin{aligned} M(f; \rho_o, f_o) &= \mathcal{F}\{m(t; \rho_o, f_o)\} \\ &= M(f - f_o) \exp\{-j2\pi f \rho_o\} \end{aligned} \quad (10.21)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Using Eqs. (10.3), (10.15), and (10.21) the right hand side of (10.20) can be evaluated. The result, for  $W^2 < 2W_q^2$ , is

$$H_{opt}(f; \rho_o, f_o) = G_1 \cdot \exp \left\{ -\pi \frac{(f - f_o \cdot \beta)^2}{W_h^2} \right\} \cdot \exp \{ +j2\pi f \rho_o \} \quad (10.22)$$

where

$$\beta = \frac{W_q^2 + \frac{W^2}{2}}{W_q^2 - \frac{W^2}{2}} \quad (10.23a)$$

$$W_h^2 = W^2 \cdot \beta \quad \text{for } \beta > 0 \quad (10.23b)$$

$$G_1 = \frac{2^{\frac{1}{4}}}{\epsilon_c} \left( \frac{1}{2} + \frac{W_q^2}{W^2} \right)^{\frac{1}{2}} \exp \left\{ +\pi \frac{f_o^2}{W_q^2 - \frac{W^2}{2}} \right\}.$$

In the event that  $W^2 = 2W_q^2$ , Eq. (10.22) becomes

$$H_{opt}(f; \rho_o, f_o) = G_2 \cdot \exp \left\{ \pi \cdot f \cdot \frac{2f_o}{W^2} \right\} \cdot \exp \{ +j2\pi f \rho_o \} \quad (10.24)$$

where

$$G_2 = \frac{2^{\frac{1}{4}}}{\epsilon_c} \cdot \left( \frac{1}{2} + \frac{W_q^2}{W^2} \right)^{\frac{1}{2}} \exp \left\{ -\pi \frac{f_o^2}{W^2} \right\} \quad (10.25)$$

The transition from the form of (10.22) to that of (10.24) will be discussed in the next section.

The frequency response function of the matched filter for the echo  $m(t; \rho_o, f_o)$  is given directly by

$$H_{mf}(f; \rho_o, f_o) = M^*(f; \rho_o, f_o) \quad (10.26)$$

$$= \frac{2^{\frac{1}{4}}}{W^{\frac{1}{2}}} \exp \left\{ -\pi \frac{(f - f_o)^2}{W^2} \right\} \cdot \exp \{ +j2\pi f \rho_o \} \quad (10.27)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

As a preliminary step toward determining the ambiguity function  $\xi(\tau, \phi)$ , Eqs. (10.3) and (10.15) yield

$$\frac{M^*(f-f_o)M(f-f_o-\phi)}{K_c(f)} = G_3 \cdot \exp \left\{ \frac{-2\pi W_q^2}{W^2 W_k^2} \left[ - \left( f_o + \frac{\phi}{2} \right) \left( 1 + \frac{W^2}{2W_q^2} \right) \right]^2 \right\} \\ \cdot \exp \left\{ \pi \left[ \hat{f}_o^2 + \hat{f}_o \hat{\phi} + \frac{1}{4} \hat{\phi}^2 \left( 1 - \frac{2W_q^2}{W^2} \right) \right] \right\} \quad (10.28)$$

where  $G_3 = \frac{2^{\frac{1}{2}} W_k}{\epsilon_c W}$  (10.29a)

$$\hat{f}_o = \frac{1}{W_q} f_o \quad (10.29b)$$

$$\hat{\phi} = \frac{1}{W_q} \phi \quad (10.29c)$$

According to Eq. (5.22), the ambiguity function is the inverse Fourier transform of the preceding expression. Using the expression for the required transform from Appendix D, one can show that

$$\xi(\tau, \phi)_{opt} = G_4 \cdot \exp \left\{ -\pi \tau^2 \frac{W^2}{2} \left( 1 + \frac{W^2}{2W_q^2} \right) \right\} \\ \cdot \exp \left\{ \pi \left[ \hat{f}_o^2 + \hat{f}_o \hat{\phi} + \frac{1}{4} \hat{\phi}^2 \left( 1 - \frac{2W_q^2}{W^2} \right) \right] \right\} \\ \cdot \exp \left\{ j2\pi \left( f_o + \frac{\phi}{2} \right) \left( 1 + \frac{W^2}{2W_q^2} \right) \tau \right\} \quad (10.30)$$

where  $G_4 = \frac{W_q}{\epsilon_c} \left( 1 + \frac{W^2}{2W_q^2} \right)$  (10.31)



The ambiguity function for the matched processor is found somewhat more simply. From Eqs. (10.3) and (5.23) one concludes that

$$\xi(\tau, \phi)_{mf} = \exp \left\{ -\pi \tau^2 \frac{W^2}{2} \right\} \cdot \exp \left\{ -\pi \frac{\phi^2}{2W^2} \right\} \cdot \exp \left\{ j2\pi \left( f_0 + \frac{\phi}{2} \right) \tau \right\}. \quad (10.32)$$

One can verify that  $|\xi(\tau, \phi)_{mf}|^2$  is in accord with the result given by Woodward<sup>49</sup> for the Gaussian pulse with  $W = 1$ .

## B. DISCUSSION OF ANALYTIC RESULTS

The major results of the analysis outlined in the preceding section will be discussed below, together with indications of their range of validity.

### 1. Signal-to-Clutter Ratio

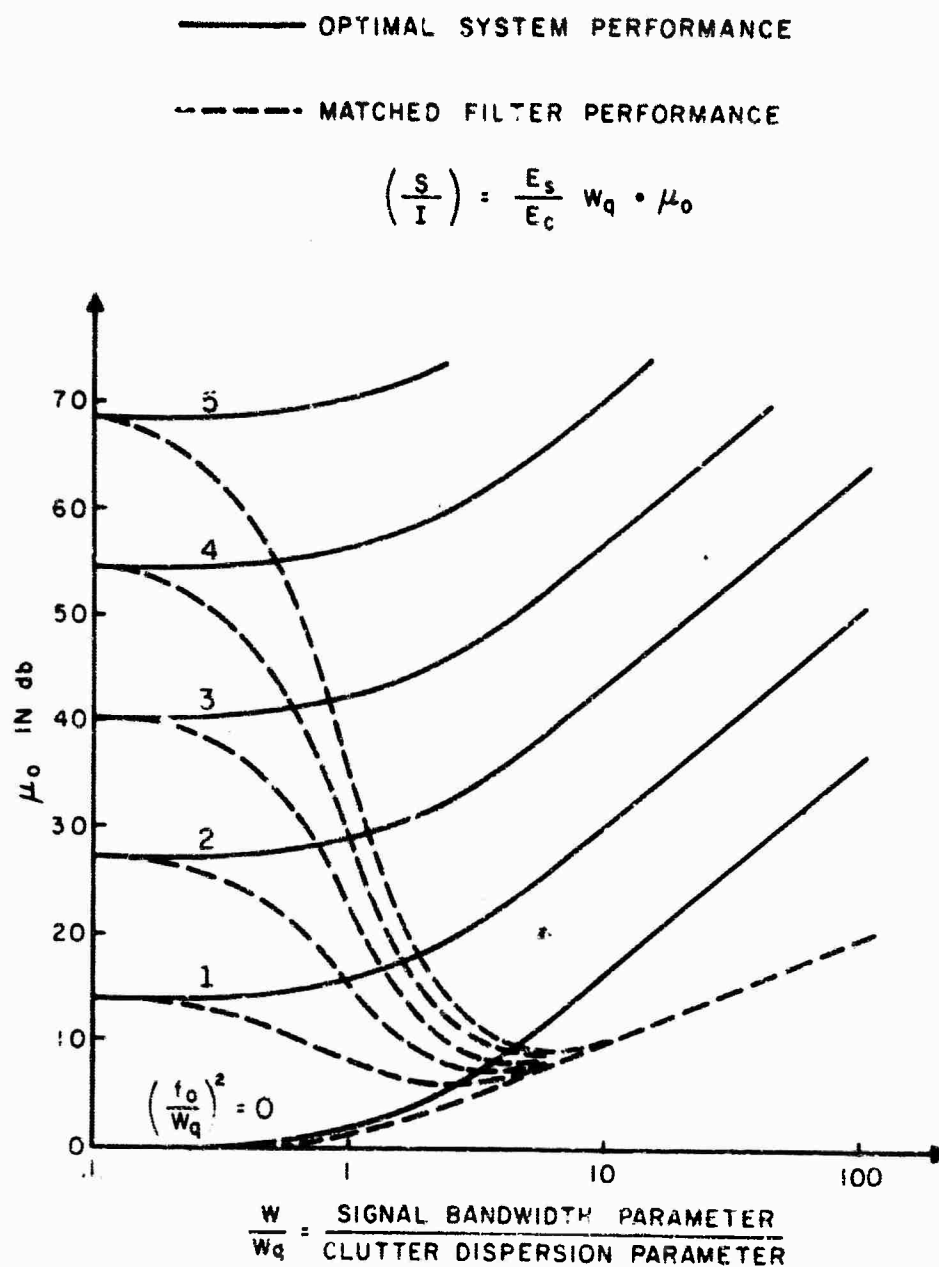
Equation (10.18) for output signal-to-interference ratio  $\frac{S}{I}$  forms the basis for Fig. 5. In terms of decibels (10.18) reads

$$10 \log \left( \frac{S}{I} \right) = 10 \log \left( \frac{\mathcal{E}_s W_q}{\mathcal{E}_c} \right) + 10 \log (\mu_0) \quad (10.33)$$

where  $10 \log(\mu_0)$  is the ordinate in Fig. 5. Along the abscissa are values of the ratio  $\frac{W}{W_q}$ . Thus, for any fixed set of parameters  $\mathcal{E}_s$ ,  $\mathcal{E}_c$ , and  $W_q$ , Fig. 5 essentially shows the variation of output signal-to-interference ratio as a function of signal bandwidth for various normalized Doppler frequencies  $\frac{f_0}{W_q}$ .

The dashed curves reflect values of  $\mu_0$  for a matched filter system and are due to Westerfield.<sup>48</sup> The solid

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES



A-194-S-0071

FIG. 5  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

curves are for  $\mu_0$  as determined in the present research for the optimal processor. For any fixed  $f_0$  it can be seen that, for all  $W_q$

$$\left(\frac{S}{I}\right)_{\text{opt}} \geq \left(\frac{S}{I}\right)_{\text{mf}} \quad (10.34)$$

as one indeed expects.

Equality of performance between the optimal and matched processors occurs only for negligibly small values of  $\frac{W}{W_q}$  at the left in Fig. 5. These correspond to signals of essentially zero bandwidth or, in the time domain, of relatively long duration. It will be seen that as the signal duration (proportional to  $\frac{1}{W}$ ) increases, the optimal processor approaches a matched processor and, consequently, the difference in performance vanishes.

As the bandwidth of the transmitted signal increases, the performance of the optimal processor improves monotonically from its zero-bandwidth value. This is in marked contrast to the rapid deterioration of matched filter performance as  $W$  becomes comparable to  $W_q$ . For  $W = W_q$  and  $f_0 = 2W_q$ , Fig. 5 indicates that optimal system performance exceeds matched filter performance by about 30 db. Moreover, as  $W$  increases above  $W_q$ , the disparity in indicated performance becomes greater still.

As already noted by Westerfield<sup>46</sup> the relatively poor performance of the matched filter is due to the relatively large overlap of the filter pass-band and the frequencies with much clutter energy. The good performance of the optimal processor on the other hand presumably stems from the effective fashion in which it tends to reject the clutter spectrum.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The mechanism for improved performance through optimal processing becomes clear upon consideration of the optimal processor frequency response function. Also clarified is the crucial role of noise in limiting system performance, particularly for values of  $W$  near and exceeding  $W_q$ .

### 2. Frequency Response Function

The frequency response function for the optimal processor is given at Eq. (10.25). Its magnitude is proportional to the Gaussian function, with

$$\text{center frequency} = f_o \cdot \beta \quad (10.35)$$

and

$$\text{bandwidth} = W_h = W \cdot \beta^{\frac{1}{2}} \quad (10.36)$$

where

$$\beta = \frac{\frac{W_q^2}{2} + \frac{W^2}{2}}{\frac{W_q^2}{2} - \frac{W^2}{2}} \quad (10.37)$$

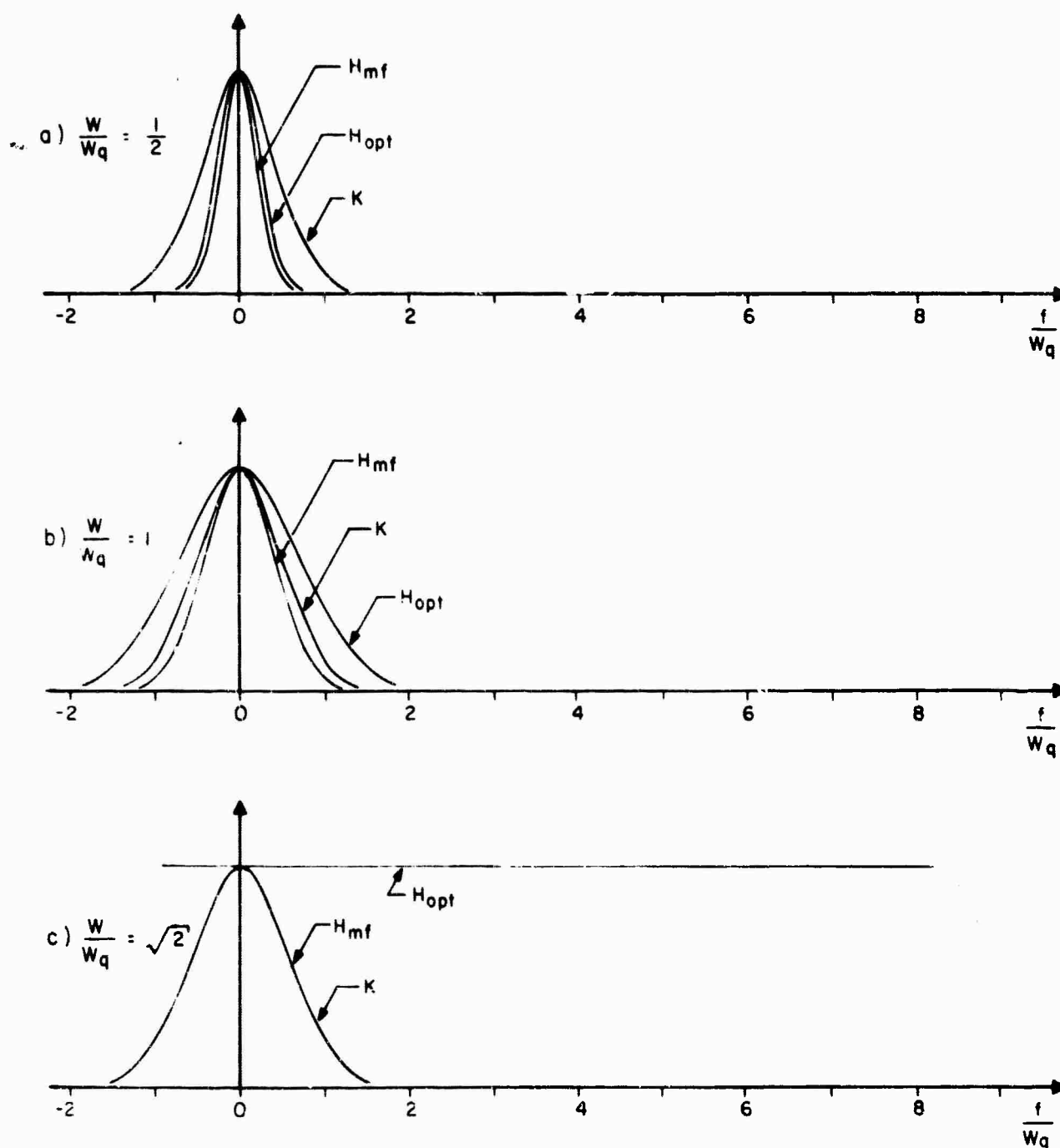
The magnitudes of the frequency response functions for optimal and matched processors are compared in Fig. 6 for three values of relative bandwidth  $\frac{W}{W_q}$ , on the assumption that an echo with no doppler shift is to be detected. A similar comparison appears in Fig. 7 where the echo to be detected has a doppler shift  $f_o$  equal to  $2W_q$ . In all cases the center frequency and bandwidth of the matched processor are  $f_o$  and  $W$  respectively.

Two conclusions may be drawn immediately, either from Eqs. (10.36) and (10.37) or from Figs. 6 and 7.

- i) The center frequencies for matched and optimal processor are different, in general; unless the echo to be detected has zero doppler frequency shift.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$f_0 = 0$$



## LEGEND:

$H_{mf}$ —MATCHED FILTER FREQUENCY RESPONSE FUNCTION

$H_{opt}$ —OPTIMAL PROCESSOR FREQUENCY RESPONSE FUNCTION

$K$  —CLUTTER INTERFERENCE POWER SPECTRAL DENSITY FUNCTION

A-194-S-0072

FIG 6 FREQUENCY RESPONSE FUNCTIONS FOR DETECTING GAUSSIAN PULSES WITH NO DOPPLER SHIFT.

$$f_0 = 2W_q$$

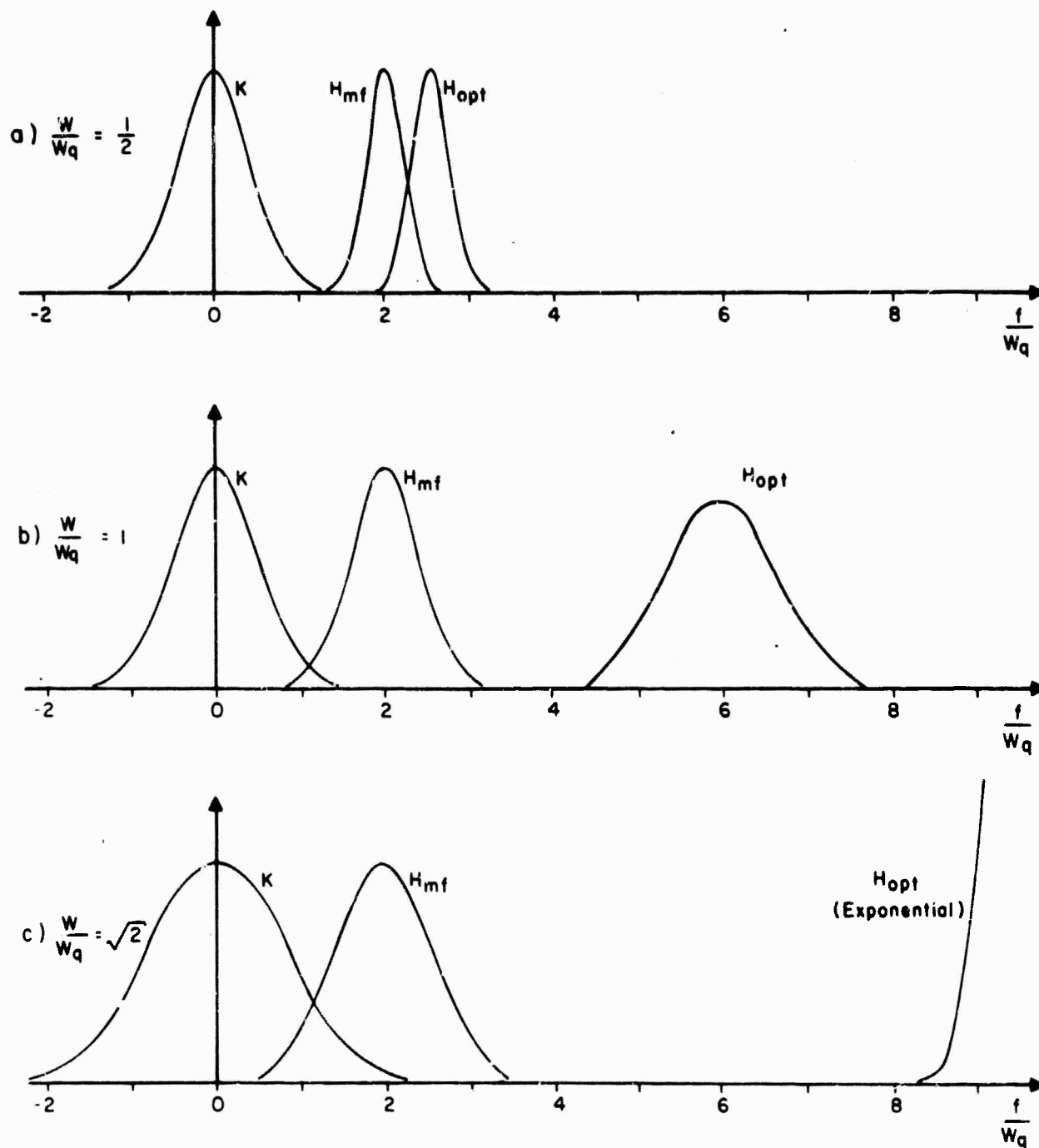


FIG 7 FREQUENCY RESPONSE FUNCTIONS FOR DETECTING GAUSSIAN PULSES WITH DOPPLER SHIFT OF  $2W_q$  cps.

- ii) The bandwidth of the optimal processor is greater than the bandwidth of the matched processor, in general; unless both are zero.

Both of these effects, in fact, become extreme as the signal bandwidth parameter  $W$  increases from zero and approaches  $\sqrt{2} W_q$ . As shown by Eq. (1.37), the factor  $\beta$  will correspondingly increase from 1 to  $\infty$ , with the striking consequences apparent in Figs. 6 and 7.

This behavior of the optimal processor is best understood by considering the formula for its frequency response function,

$$H(f; \rho_o, f_o) = \frac{M^*(f; \rho_o, f_o)}{K_c(f)} \quad (10.38)$$

In the present case, when noise is ignored, both numerator and denominator are Gaussian functions, with the result that the quotient is also Gaussian. For small signal bandwidths, however, the bandwidth of the numerator is less than the bandwidth of the denominator. Therefore as  $f$  becomes large the quotient goes to zero.

However, for  $W^2 = 2W_q$ , the numerator and denominator attain equal bandwidths. For this case, if  $f_o = \hat{\phantom{f}}$ , the result is that numerator and denominator are identical Gaussian functions, and the quotient is independent of frequency. Indeed, if  $W^2$  exceeds  $2W_q$  the numerator of (10.38) does not fall off with increasing frequency as rapidly as the denominator. The result is that the quotient, i.e.,  $H(f; \rho_o, 0)$ , diverges to infinity with increasing  $f$ .

The results given in Figs. 5, 6, and 7 were derived on the assumption that noise was negligible and clutter was the only interference. Examination of Figs. 6 and 7, however, suggests that the effects of noise cannot be so easily ignored.

In the first place, let it be noted that the mean-square noise interference at the output of any of the optimal processors discussed will be greater than for the corresponding matched processors, because of the difference in optimal and matched bandwidths. In fact one expects that for any noise level, no matter how small, the output noise will at some point exceed the output clutter as the bandwidth  $W_h$  approaches infinity.

One concludes that the optimal processor performance implied by Fig. 5 for value of  $\frac{W}{W_q}$  exceeding  $\sqrt{2}$  will be fundamentally unattainable because of the unavoidable presence of noise at the processor input.

This is not to say that a processor which is appropriately optimized for the clutter-plus-noise interference now being discussed cannot be significantly better in performance than a matched filter. Rather, the possible effects of noise must be realistically assessed.

A case in point is the seeming performance advantage of 30 db which has already been mentioned for the optimal processor when  $W = W_q$  and  $f_o = 2W_q$ . In this case the bandwidth factor  $\sqrt{\beta}$  is only  $\sqrt{3}$  so that the optimal processor bandwidth is not markedly greater than the matched processor bandwidth. One does not therefore, at first, expect that noise will be a significant factor in system performance. This case nevertheless indicates a second aspect of the problem.

† The optimal processors for clutter alone are, of course, no longer optimal for the mixed clutter-plus-noise interference being discussed.



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The processor yielding the results of Fig. 5 was taken to be

$$H_1(f; \rho_o, f_o) = \frac{M^*(f; \rho_o, f_o)}{K_c(f)} \quad (10.39)$$

whereas in the presence of noise the optimal frequency response function is really

$$H_2(f; \rho_o, f_o) = \frac{M^*(f; \rho_o, f_o)}{K_c(f) + \eta_o} \quad (10.40)$$

where  $\eta_o$  is the white noise power spectral density. The question is whether  $H_1(f; \rho_o, f_o)$  is an adequate approximation to  $H_2(f; \rho_o, f_o)$ . The answer will be a qualified "yes," if

$$K_c(f) \gg \eta_o \quad (10.41)$$

for those frequencies  $f$  where  $H_1(f; \rho_o, f_o)$  has significant values.

It has already been learned however that the frequencies in question are not necessarily the same as the echo center frequency  $f_o$ . Rather, the processor center frequency has already been identified as  $f_o \cdot \beta$ , so that (10.41) might be replaced by

$$K_c(f_o \cdot \beta) \gg \eta_o \quad (10.42)$$

The significant point is that  $f_o \cdot \beta$  and the nearby larger frequencies can lie relatively far away from the frequencies where most of the clutter energy is concentrated. When  $f_o = 2W_q$  and  $W = W_q$ , for example, the value of  $\beta$  is 3 and

$$f_o \cdot \beta = 6W_q \quad (10.43)$$

As can be seen in Fig. 7,  $K_c(6W_q)$  is indeed small relative to  $K_c(0)$ . The figure has no general significance, but for the Gaussian function in the present case

$$\frac{K_c(6W_q)}{K_c(0)} = -300 \text{ db} \quad (10.44)$$

One concludes that for an optimal processor to have characteristics determined essentially by the clutter interference, without consideration of noise, the noise energy must be negligible with respect to the clutter energy, even at those frequencies where a possibly minor part of the clutter energy lies.

In the present case it so happens that the signal energy is also small at  $f = 6W_q$ . The result is that the performance implied by Fig. 5 for  $f_o = 2W_q$  and  $W = W_q$  will not be achieved by the processor of Fig. 7b unless the noise spectral density is negligible with respect to both the relatively small signal and clutter energies in the vicinity of  $f = 6W_q$ .

### 3. Ambiguity Function

The major features of the ambiguity function for the optimal processor may be surmised after careful study of Figs. 6 and 7. They are readily extracted, however, from Eq. (10.30). If attention is confined to the doppler resolution profile for zero delay mis-alignment, i.e.,  $\tau = 0$ , then one can find with little algebra that

$$|\xi(0, \phi)|_{\text{opt}} \propto \exp \left\{ -\pi \frac{(\hat{\phi} - \hat{\phi}_o)^2}{\hat{W}_\phi^2} \right\} \quad (10.45)$$

where

$$\hat{\phi}_o = \hat{f}_o / 2 \left( \frac{2W_q^2}{W^2} - 1 \right) \quad (10.46a)$$

and

$$\hat{W}_\phi = 2 / \left( \frac{2W_q^2}{W^2} - 1 \right)^{\frac{1}{2}}.$$

In the transition case, when  $W^2 = 2W_q^2$ , one concludes directly from (10.30) that

$$|\xi(0, \phi)|_{\text{opt}} \propto \exp \{ \pi \cdot \hat{f}_o \cdot \hat{\phi} \} \quad (10.47)$$

Equations (10.45) and (10.47) may therefore be plotted as in Figs. 8 and 9 for the same sets of parameters appearing in Figs. 6 and 7. The well-known doppler profile of the matched processor ambiguity function for a Gaussian pulse is simply (from 10.32)

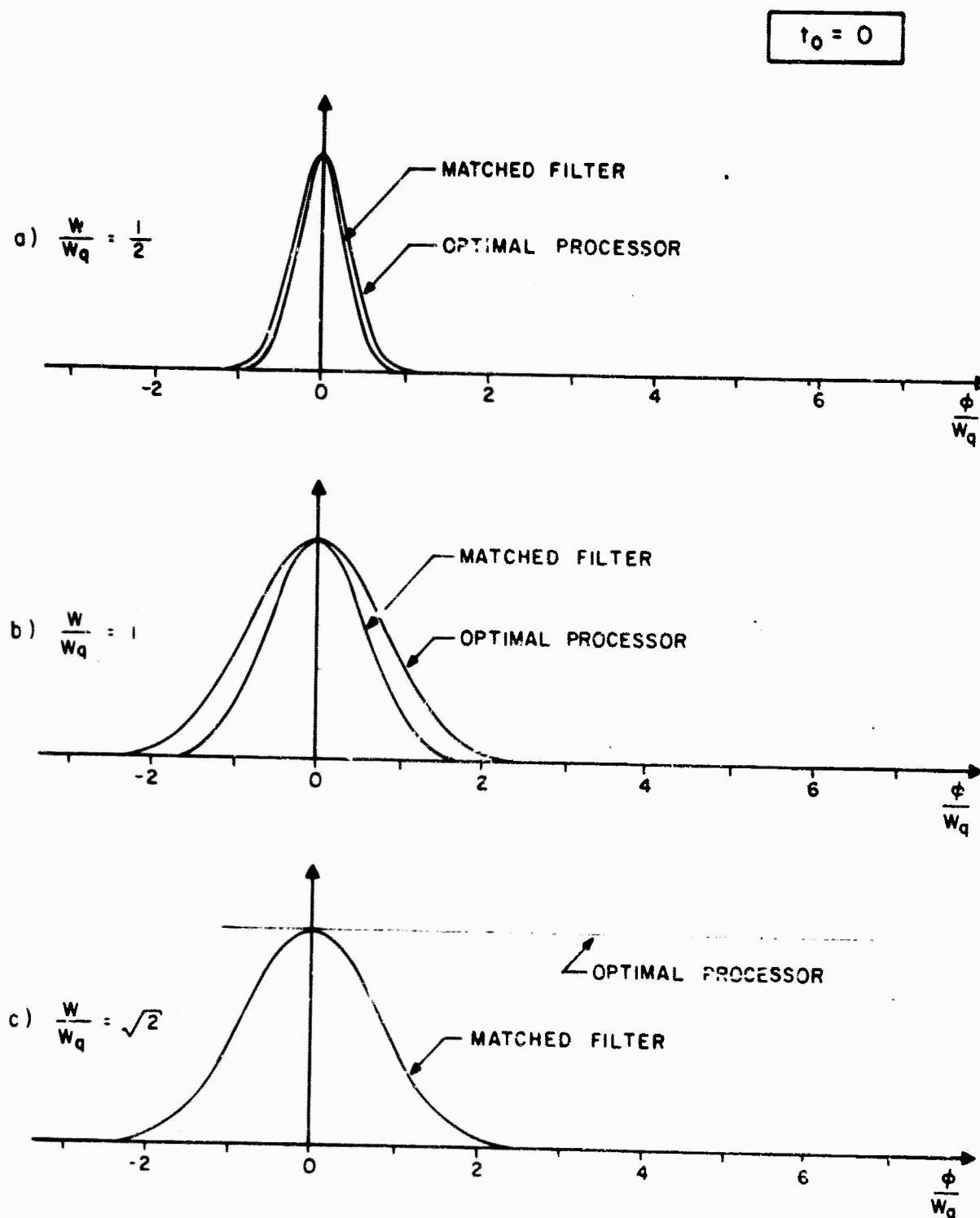
$$| \xi(0, \phi) |_{mf} \propto \exp \left\{ -\pi \hat{\phi}^2 \cdot \frac{W_q^2}{2W^2} \right\} \quad (10.48)$$

where the normalized  $\hat{\phi}$  has been introduced.

The matched filter ambiguity function is independent of the doppler frequency  $f_o$  of the echo to be detected. It is characteristically symmetrical, in the  $\tau = 0$  plane, about  $\phi = 0$ . The optimal processor ambiguity function, on the other hand, depends upon  $f_o$  as well as  $\phi$  and is distinctly unsymmetrical about  $\phi = 0$ .

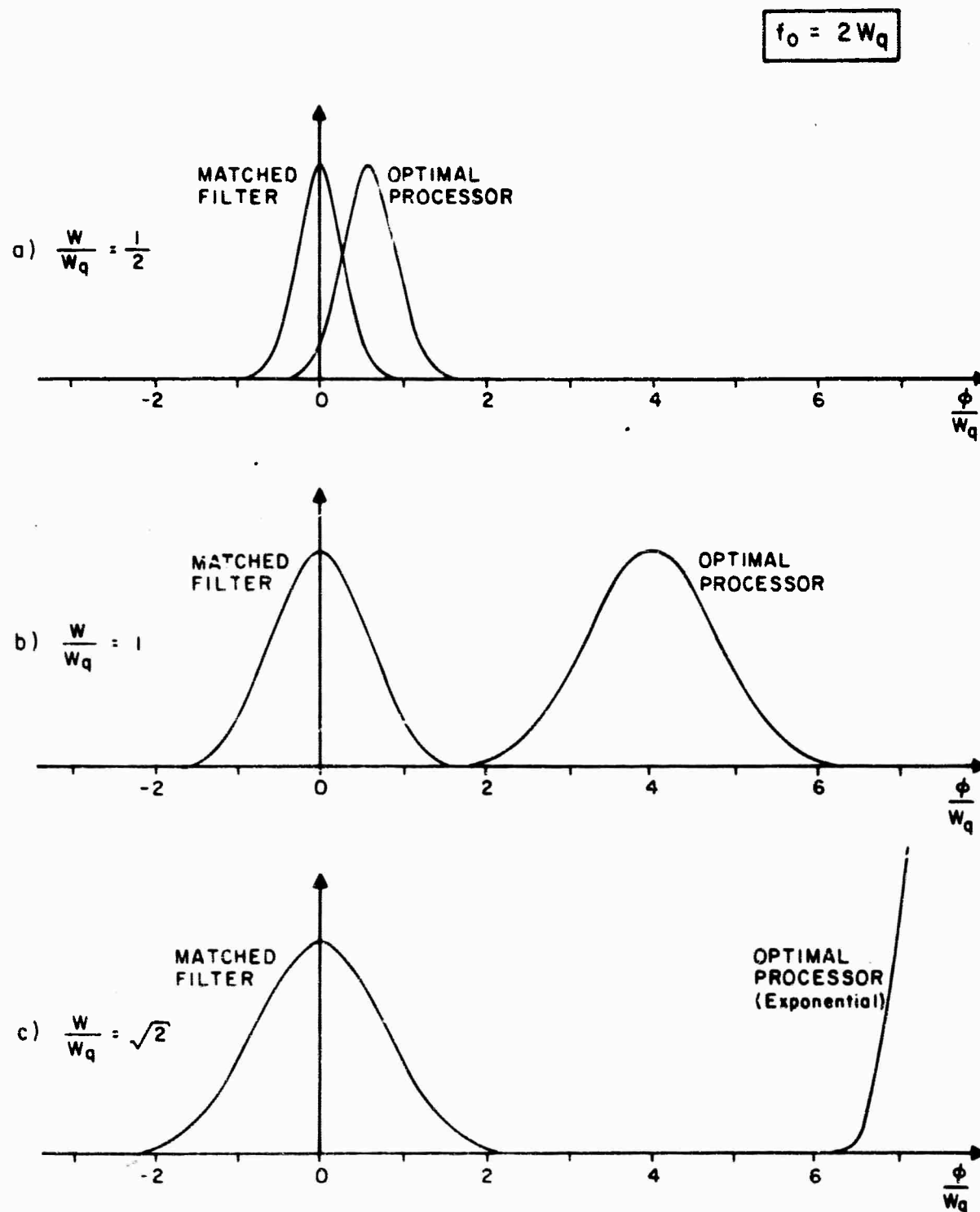
The asymmetry of the optimal ambiguity function arises directly out of the asymmetry of the optimal frequency response function with respect to the echo spectrum. The relative maximum of  $| \xi(0, \phi) |_{opt}$  occurring at  $\phi = 4W_q$  in Fig. 9b, for example, directly reflects the fact that the pass-band for  $H_{opt}(f)$  lies  $4W_q$  cps above the signal spectrum located at  $2W_q$  cps, as shown in Fig. 7b. Signals with a doppler shift of  $(2W_q + 4W_q)$  cps therefore yield a greater processor output than the "design" signal of doppler shift  $2W_q$  cps.

Because of the indicated asymmetry, the ambiguity function which is "optimal" from the view-point of clutter suppression is not necessarily even "desirable" from other possible viewpoints. An ambiguity function such as that of Fig. 9b would be eminently undesirable in a system which was required to provide an indication of echo doppler shift  $f_o$ ,



A-194-S-0074

FIG. 8 DOPPLER PROFILES OF AMBIGUITY FUNCTIONS FOR DETECTING GAUSSIAN PULSES WITH NO DOPPLER SHIFT.



A-194-S-0075

FIG. 9 DOPPLER PROFILES OF AMBIGUITY FUNCTIONS FOR DETECTING GAUSSIAN PULSES WITH DOPPLER SHIFT OF  $2W_q$  cps.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

in addition to simply detecting the presence of the echo. The presence of a weak signal with doppler shift  $6W_q$  for example would not be distinguishable from, and could be mistaken for, the presence of a signal with the "design" doppler shift  $2W_q$ .

For applications requiring doppler frequency estimation the symmetric ambiguity functions associated with matched filter receivers are attractive and provide, not unreasonably, the basis for system design. To attempt to maintain an approximate symmetry, while simultaneously attempting to reduce clutter interference through appropriate choice of a processor weight function, is to strive for "good" system performance according to two possibly contradictory criteria. As an analytic problem, it was not attempted in this research. As an intuitive problem it retains the difficulties originally discussed in connection with, and following, equation (5.9).

Figures 7b and 9b illustrate, as well as any figures, the possible difficulty of an intuitive problem approach. The task might be undertaken of modifying the known matched filter transfer function of Fig. 7b, or the matched ambiguity function of 9b, in order to reduce clutter. To reduce the transmission of frequency components between  $f = 0$  and, say  $f = f_0$  would be a natural endeavor. It would hardly be natural or intuitive, however, to expect that the "best" modification of the matched filter would be, essentially, to translate it upward in frequency by  $4W_q$  cps and to increase its bandwidth somewhat.

Two other attributes of the optimal solutions presented in Figs. 6 through 9 also deserve comment. First of all, the particular functional forms (in this example, all Gaussian) depend strictly upon the assumed modulation function and clutter dispersion function. Alteration of either of these

## COLUMBIA UNIVERSITY--ELECTRONICS RESEARCH LABORATORIES

Functions can lead to solutions greatly different in detail from the present ones†. In a very real sense, therefore, any optimal processor is optimal only for a particular set of signal, noise, and clutter circumstances. A processor with performance not greatly dependent upon the particular interference at its input would be desirable, even if not strictly optimal for most input conditions.

The second attribute of the Gaussian optimal processors is a certain general similarity to the clutter rejection systems yielding "moving target indication," (MTI). In MTI systems,†† echo components with small doppler shifts lie in the region of strong clutter energy and are suppressed. Echo components of larger doppler shift are more or less uniformly amplified to give the system output. Moreover, the purpose of an MTI system is merely to indicate those targets which are moving, without giving an indication of their velocity.

The transitional solutions of Figs. 7c and 9c have these general characteristics of an MTI system. Signals of small doppler shift are suppressed by the exponential frequency characteristic while signals of larger doppler shift are amplified. This optimal processor would also give only an output indication, with no estimate of velocity, for signals of large enough doppler shift, just as an MTI system.

The similarities between optimal and MTI systems have a common origin, in that both systems act to suppress frequency components containing relatively great clutter energy while emphasizing the remainder of the received spectrum.

---

† A frequency response function with poles located periodically along the frequency axis, for example, appears in an early paper<sup>45</sup> on clutter reduction.

†† See, for example, References 39 and 47.

C. NUMERICAL RESULTS (FOR CLUTTER PLUS NOISE)

The analytic results of section A of this chapter are, in principle, quite readily modified to include the effect of "white" noise added to the Gaussian clutter interference. Thus Eq (10.17) for the optimal signal-to-interference ratio becomes

$$\frac{S}{I} = 2\mathcal{E}_s \cdot \int_{-\infty}^{\infty} \frac{|M(f; \rho_o, f_o)|^2}{K_c(f) + N_o} df \quad (10.49)$$

(see eqs. 4.49 and 4.51), while eq. (10.20) for the optimum frequency response function becomes

$$H_{opt}(f; \rho_o, f_o) = W^*(f; \rho_o, f_o) = \frac{M^*(f; \rho_o, f_o)}{K_c(f) + N_o} \quad (10.50)$$

where  $K_c$  is the noise power spectral density in watts per cycle per second. The additional presence of  $N_o$  in each of these denominators, however, prevents the realization of simpler analytic expressions than Eqs. (10.49) or (10.50) for either  $\frac{S}{I}$  or  $H_{opt}(f; \rho_o, f_o)$ . For actual values of either of these quantities, therefore, and for  $\frac{S}{I}$  especially, numerical calculations are necessary.

1. Signal-to-Interference Ratio

Fortunately the integration indicated in Eq. (10.49) for  $\frac{S}{I}$  can be approximated quite well, and with comparatively little effort, by the Hermite - Gauss quadrature formula.<sup>†</sup> This numerical method has therefore been used in this research to compute  $\frac{S}{I}$  for various values of signal bandwidth  $W$ ,

<sup>†</sup> See Hildebrand<sup>16</sup>, pp. 319 to 330, or Appendix E of this dissertation.



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

echo doppler frequency  $f_o$ , and noise level  $N_o$ , for fixed signal and clutter levels. The resulting data then yields the graphs presented as Figs. 10b through 10h.

It will be recalled that the clutter-to-noise ratio parameter  $R_c$  was defined as (see Eq. 9.19)

$$R_c = \frac{2\hat{\mathcal{E}}_c}{N_o} \quad (10.51)$$

where 
$$\hat{\mathcal{E}}_c = \max_{(\rho, f)} \mathcal{E}(\rho, f) \quad (10.52)$$

In the present circumstances one may verify, using Eqs. (10.7) and (10.8), that

$$\hat{\mathcal{E}}_c = \frac{\mathcal{E}_c}{W_q} \quad (10.53)$$

and that consequently

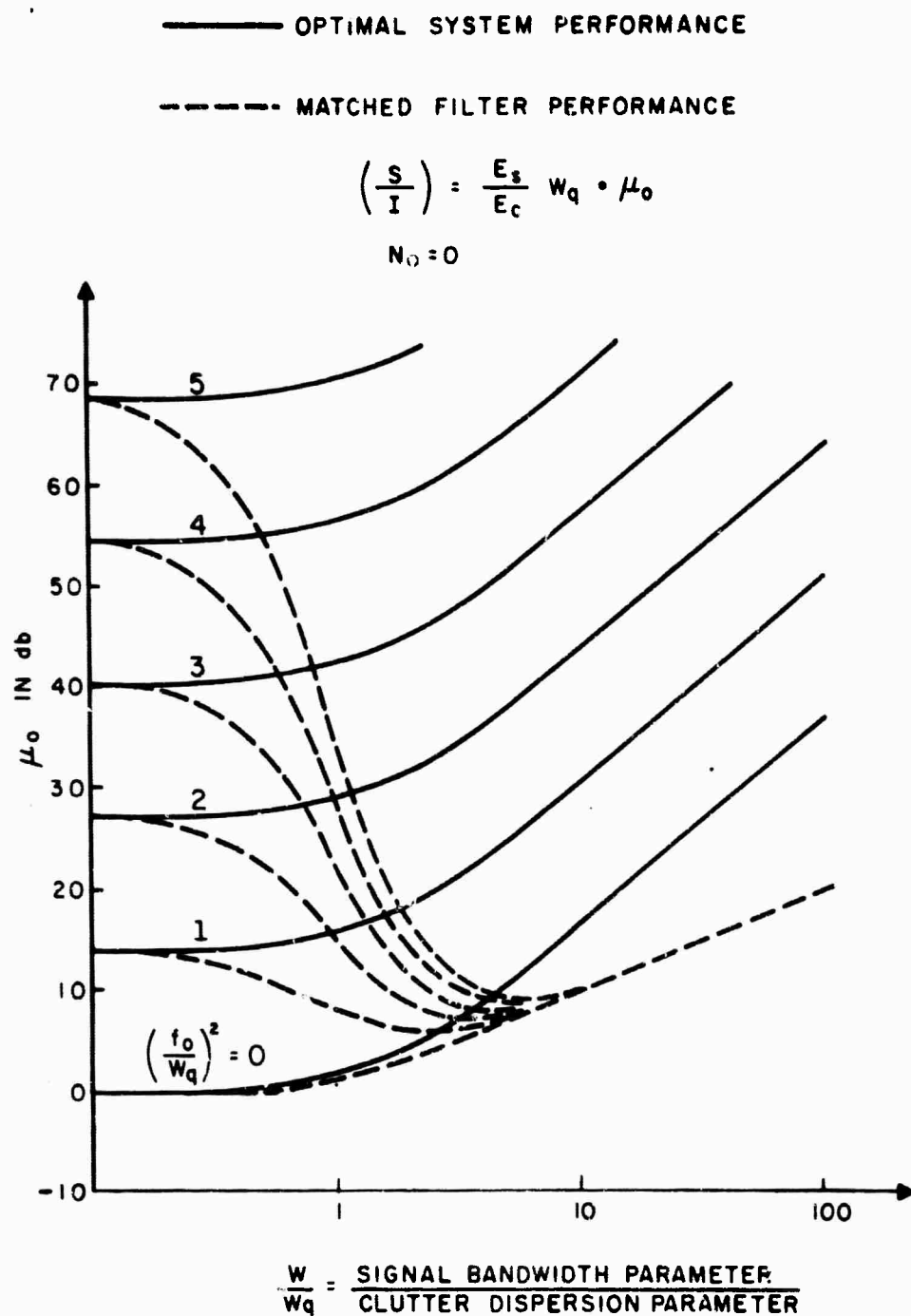
$$R_c = \frac{2\mathcal{E}_c}{W_q N_o} \quad (10.54)$$

It may also be verified, using Eqs. (10.13) and (10.15), that

$$K_c(0) = \frac{2\mathcal{E}_c}{\left(W_q^2 + \frac{W^2}{2}\right)^{\frac{1}{2}}} \leq \frac{2\mathcal{E}_c}{W_q} \quad (10.55)$$

Since  $K_c(f)$  reaches its maximum value at  $f$  equal to zero, one sees that  $\frac{2\mathcal{E}_c}{W_q}$  corresponds to the maximum clutter-

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES



A-194-S-0071

FIG. 10a  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS AND  $R_c = \infty$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

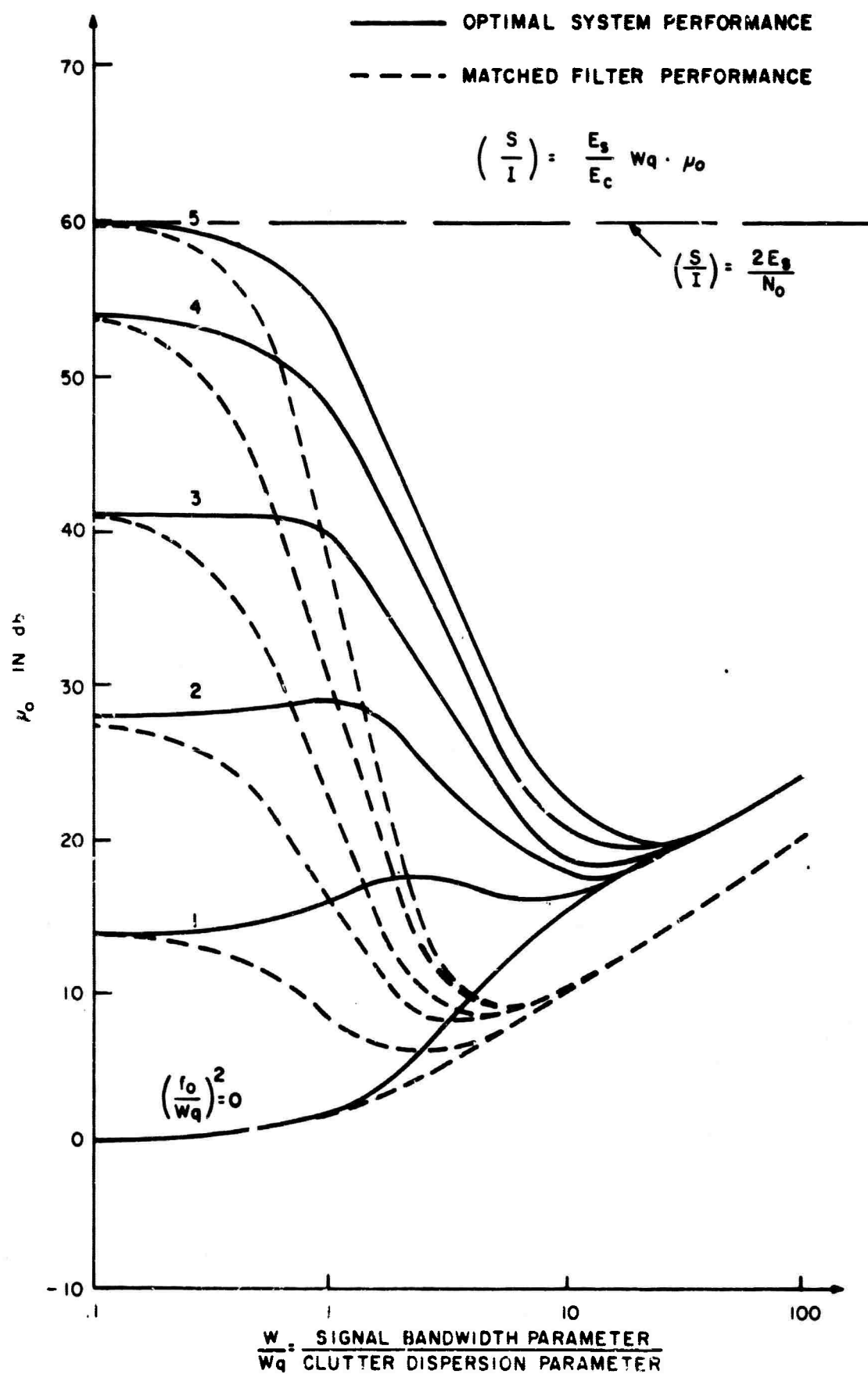


FIG. 10b  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS AND  $R_c = 60$  db

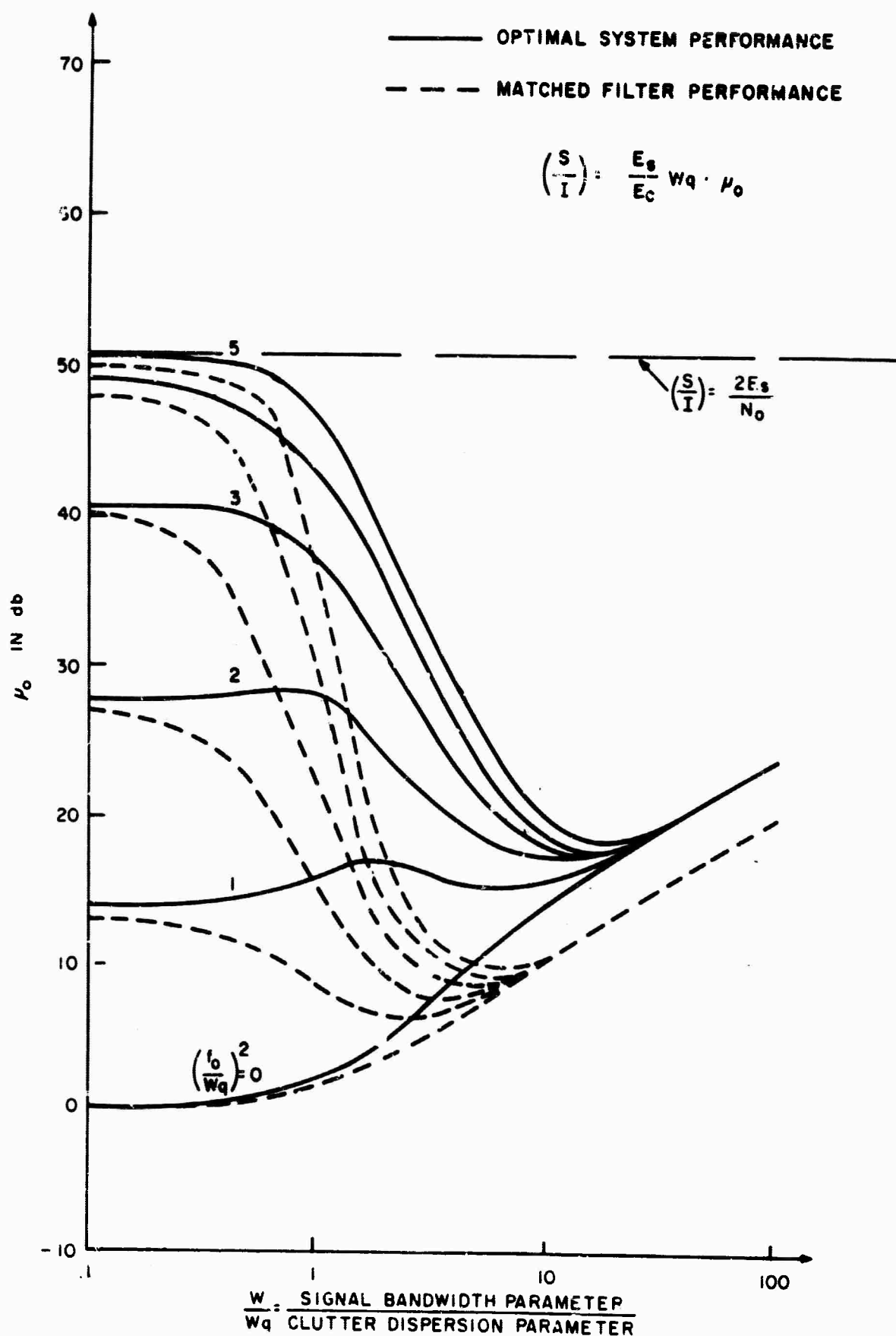


FIG.10c  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS AND  $R_c = 50$  db

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

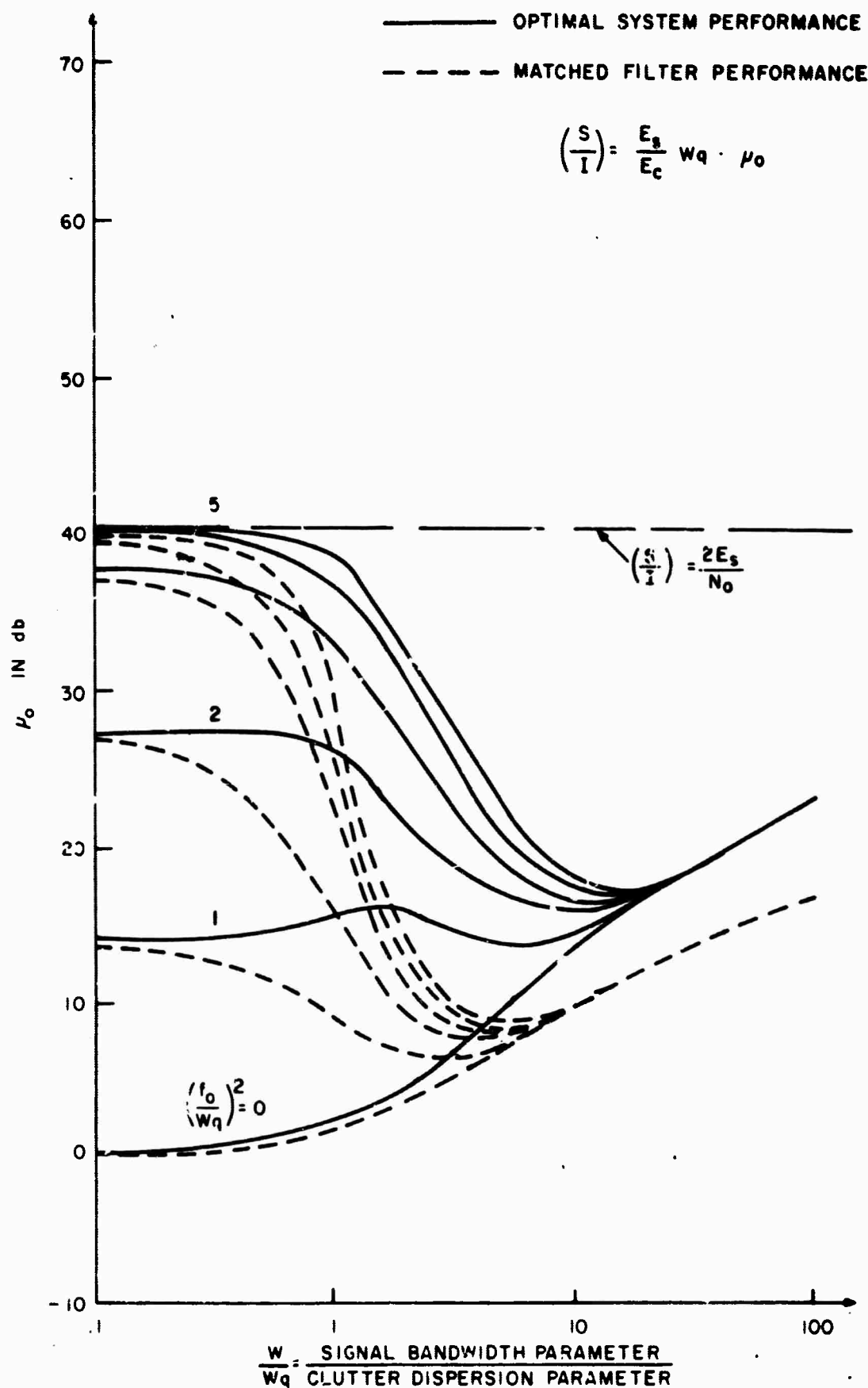


FIG. 10d  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS AND  $R_c = 40\text{db}$

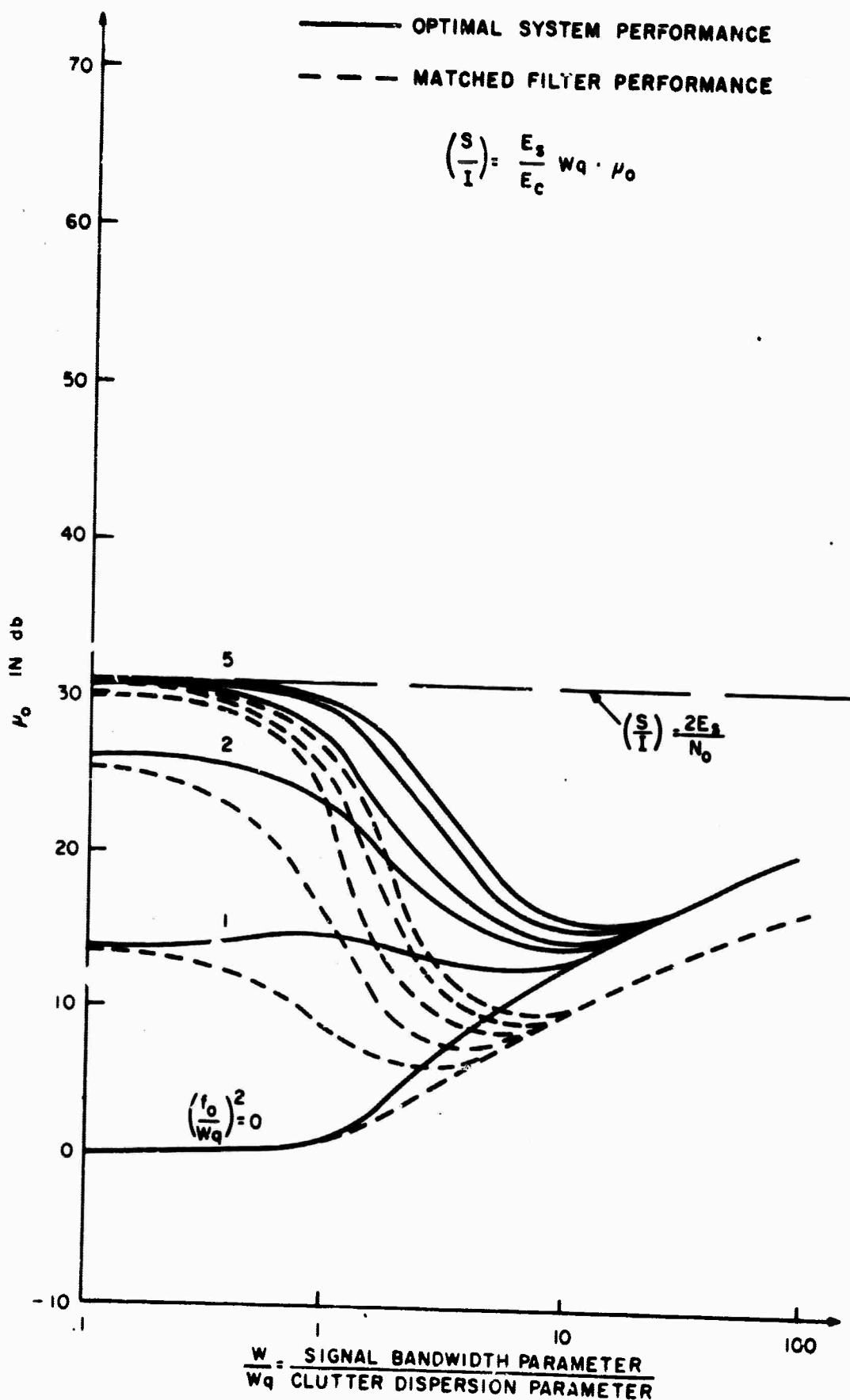


FIG. 10e  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS AND  $R_c = 30\text{db}$

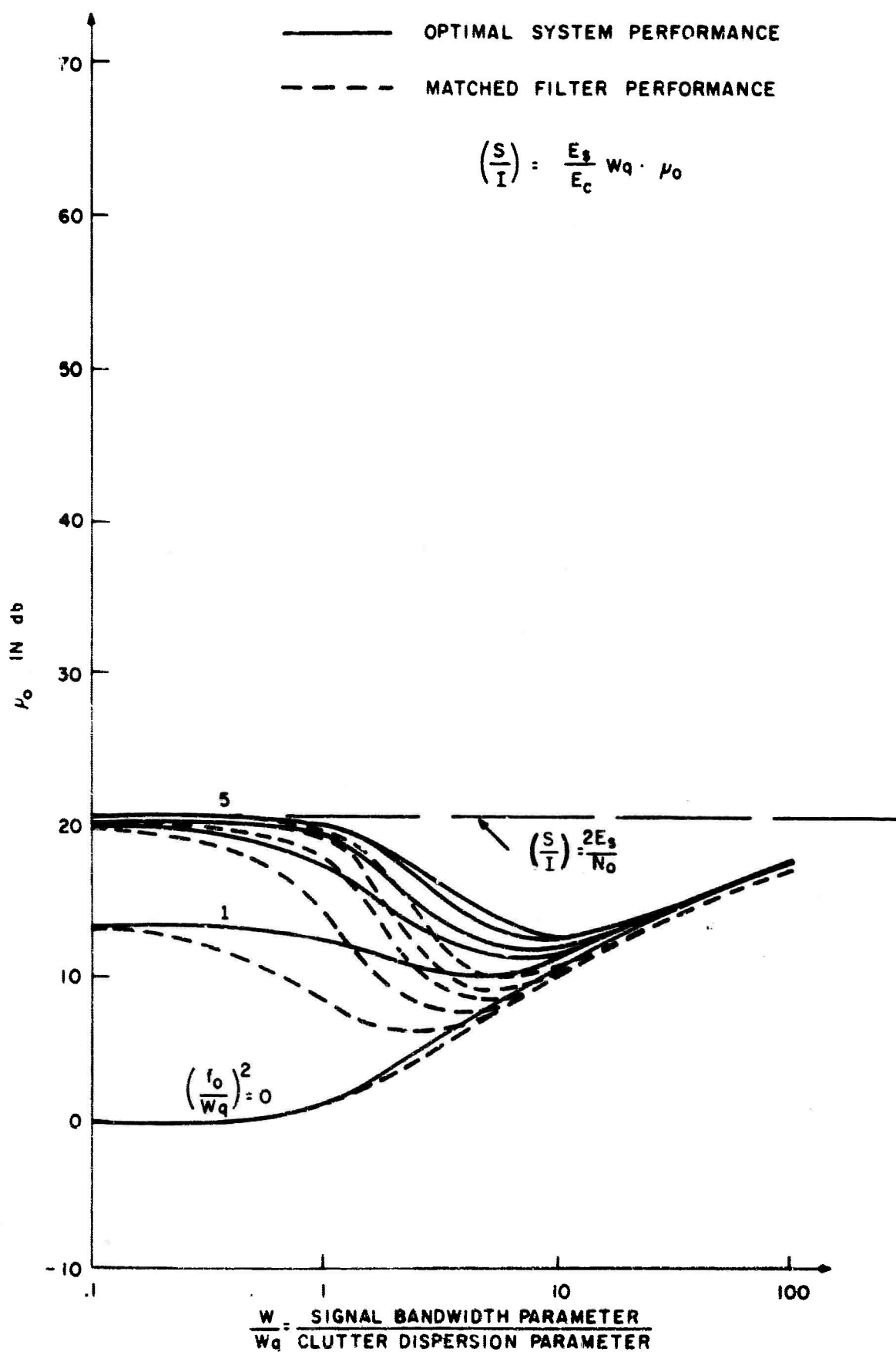


FIG. 10f  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS AND  $R_c = 20\text{db}$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

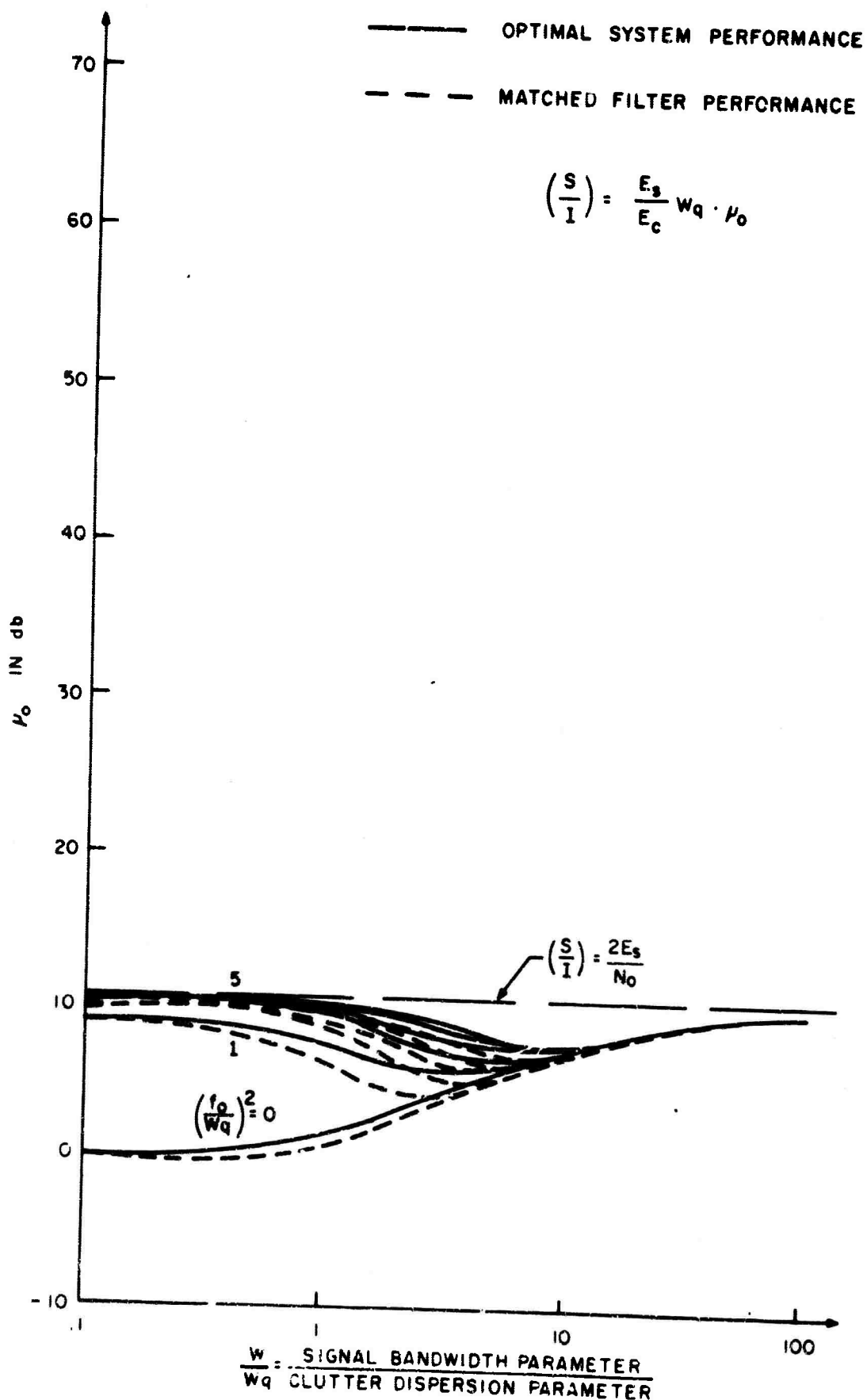


FIG. 10  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS AND  $R_c = 10$ db



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

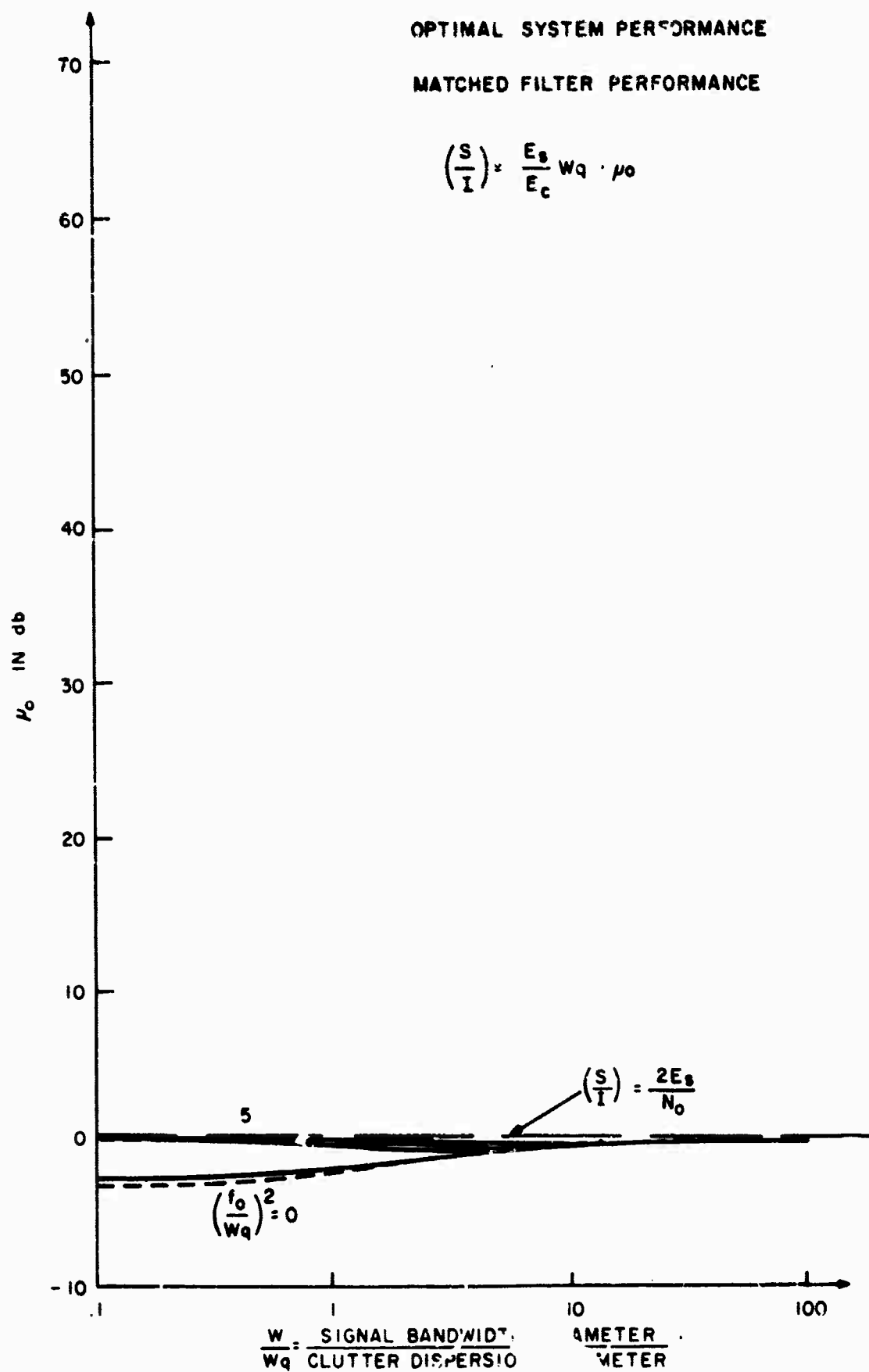


FIG. 10h  $\mu_0$  AS A FUNCTION OF SIGNAL BANDWIDTH FOR VARIOUS DOPPLER FREQUENCY SHIFTS AND  $R_c = 0$  db

ter power spectral density achieved for any transmitted (Gaussian) waveform. Thus,  $R_c$  is a measure of the clutter-to-noise power ratio for the largest clutter components near zero frequency. Unless  $W^2 \gg W_q^2$ , a value of  $R_c$  equal to zero decibels therefore means that, in the vicinity of  $f = 0$ , clutter and noise powers are comparable. If  $W = 0$ , then  $R_c$  is exactly the clutter-to-noise ratio at  $f = 0$ .

The value of  $R_c$  equal to 60 db, indicates a clutter-to-noise ratio of about 60 db for  $f = 0$ . The corresponding noise level is therefore reasonably described as "small" compared to the clutter level.

Figure 10a is a repetition of Fig. 5 for convenient comparison in the present context. It presents a comparison of the ratios  $\frac{S}{I}$  achieved by optimum and matched processors for several doppler shifts  $f_o$ , as a function of echo bandwidth  $W$  in the absence of noise. Figures 10b through 10h present similar data for increasing noise levels  $N_o$  (or decreasing  $R_c$ ).

The most striking aspect of this collection of performance data is the great difference between Fig. 10a and the other Figs. 10b through 10h. The very great differences between  $(\frac{S}{I})_{opt}$  and  $(\frac{S}{I})_{mf}$  which would exist for large bandwidths  $W$  in the absence of noise, in Fig. 10a, are seen to be almost entirely eliminated by the presence of even the very small noise level for Fig. 10b.

This result is not unexpected in view of the discussion in the preceding section. Its cause, in terms of the effects of noise upon the optimum frequency response function, will be discussed presently. At the moment it suffices to observe that the presence of even a small noise

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

level has greatly modified optimum system performance. It may also be verified incidentally that matched processor performance does not differ greatly between Figs. 10a and 10b.

When any noise is present the performances of both optimum and matched processors are seen to have many similar features. In the first place the increasing noise level represented by the progression of figures from 10b to 10h is seen to cause a (totally expected) general lowering of the plotted curves, corresponding to decreasing signal-to-interference ratios.

In the Figs. 7b through 7h a horizontal dashed line has been drawn to indicate the level for  $\mu_0$  which corresponds to

$$\frac{S}{I} = \frac{2\mathcal{E}_s}{N_0} \quad (10.56)$$

This performance would not be exceeded by any linear processor acting in the presence of noise alone. As shown in Figs. 10b through 10h, it is certainly not exceeded when the additional clutter interference is present.

Subject to the limit imposed by equation (10.56), performance is seen to improve as  $f_0$ , the doppler frequency difference between the echo and the clutter mean doppler frequency, increases.

In all cases three bandwidth regions can be seen for both optimum and matched processors. For narrow echo bandwidths ( $\frac{W}{W_g} \leq 1$ ), both processors have essentially the same performance at all noise levels. The performance in-

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

deed is limited only by the noise level, as most of the clutter doppler spectrum is rejected by the processor.

For bandwidths  $W$  approximately equal to the clutter dispersion bandwidth  $W_q$ , the performances of both processors enter a transition region of generally declining  $\frac{S}{I}$  ratios with increasing bandwidth. It is in this transition region that the greatest differences between optimum and matched performances are to be seen.

Beyond the transition region, for  $\frac{W}{W_q} \geq 10$  say, both processors exhibit a gradually increasing  $\frac{S}{I}$  ratio, attributable to the increasing range resolution associated with increasing bandwidth. In this asymptotic region, the clutter power level at the processor input is decreasing linearly with increasing signal bandwidth. This results in the linearly increasing output  $\frac{S}{I}$  ratios shown in Figs. 10b through 10h. The upward trend of the  $\frac{S}{I}$  ratio with bandwidth is, of course, ultimately checked by the constraint of Eq. (10.56). This leveling off of  $\mu_o$  for large  $\frac{W}{W_q}$  is evident in Fig. 10g for  $f_o = 1 \cdot W_q$ , say, and  $W$  equal to  $100 W_q$ .

### 2. Frequency Response Functions

Typical frequency response functions for the optimum processors leading to the graphed performance data of Figs. 10 have been computed from Eq. (10.50) for various noise levels. A number of these frequency response functions are shown in Figs. 11 and 12, for echo doppler frequency shifts of zero and  $2W_q$  respectively. These two figures, for non-zero noise, correspond to Figs. 6 and 7 respectively for the zero-noise case. Where clarity permits, the zero-noise optimum frequency response functions are also

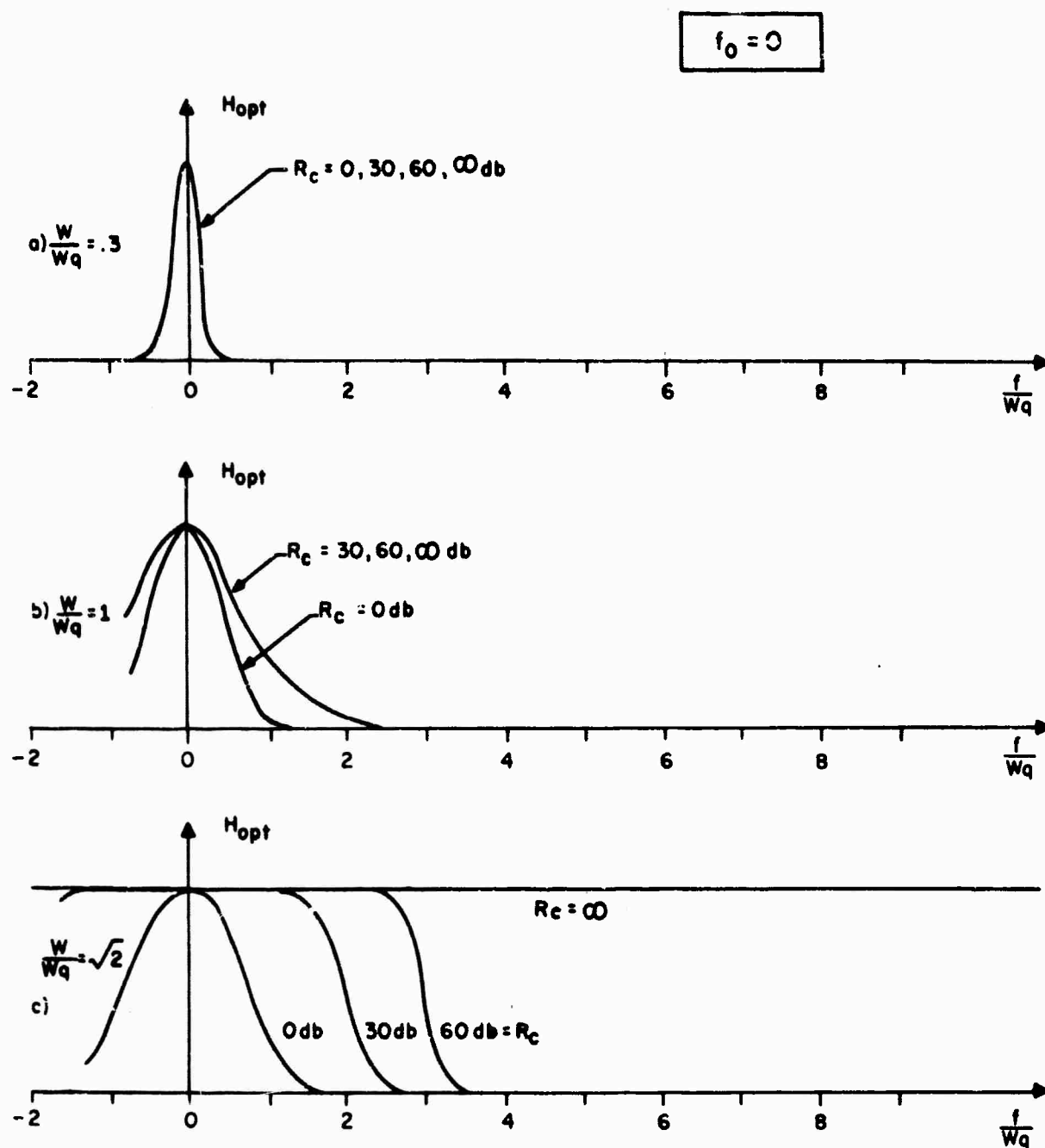


FIG. II FREQUENCY RESPONSE FUNCTIONS FOR DETECTING GAUSSIAN PULSES WITH NO DOPPLER SHIFT IN VARIOUS CLUTTER LEVELS

$$f_0 = 2Wq$$

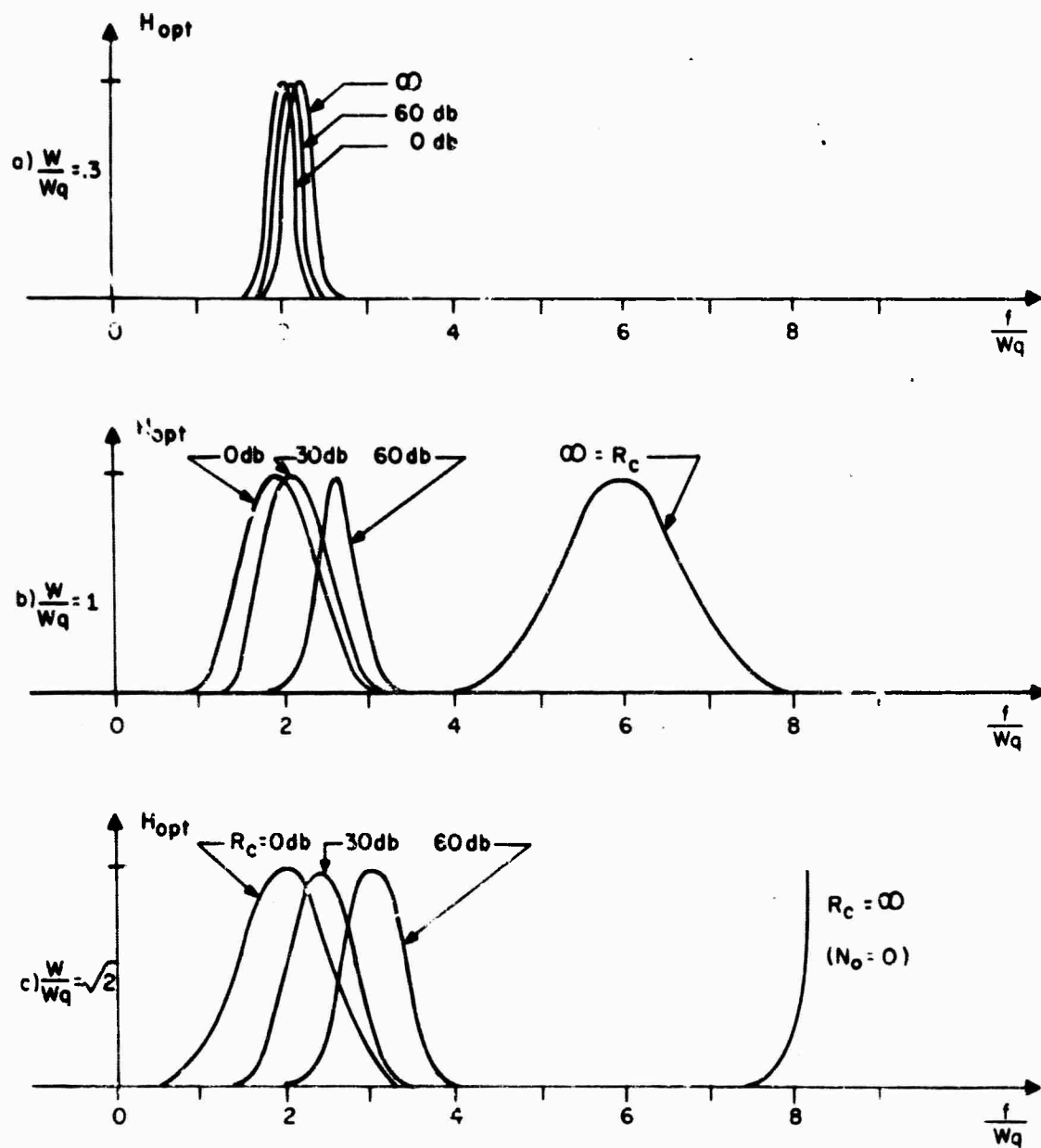


FIG 12 FREQUENCY RESPONSE FUNCTIONS FOR DETECTING GAUSSIAN PULSES WITH DOPPLER SHIFT OF  $2Wq$  cps IN VARIOUS NOISE LEVELS

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

shown in Figs. 11 and 12. The matched frequency response functions  $M_{mf}(f; \rho_o, f_o)$  and the clutter power spectral density functions  $K(f)$ , however, do not depend upon the noise level and are available from Figs. 6 and 7.

The curves of Figs. 11 and 12 correspond to noise levels in eqs. (10.49) and (10.50) which yield

$$\mathcal{R}_c = \infty, 60 \text{ db}, 30 \text{ db}, \text{ and } 0 \text{ db}, \quad (10.57)$$

Apart from the no-noise case this spans a range of 60 db, from noise relatively small compared to the clutter ( $\mathcal{R}_c = 60 \text{ db}$ ) to noise of a level comparable to the clutter level ( $\mathcal{R}_c = 0 \text{ db}$ ).

For small echo bandwidths, the optimum frequency response function is essentially unaffected by the noise level (see Figs. 11a and 12a). As in the no-noise case  $H_{opt}(f; \rho_o, f_o)$  and  $H_{mf}(f; \rho_o, f_o)$  are essentially identical for  $W \leq 0.1 W_q$ . This is consistent with the similarity of performance in this range which has already been noted.

It is for the intermediate and larger echo bandwidths that the effects of noise are strikingly evident.

Recall the case for  $f_o = 2W_q$  and  $W = W_q$  which was shown in Fig. 6b, where detection of an echo with doppler shift  $2W_q$  required an optimum frequency response with pass-band centered about  $6W_q$ . From Fig. 12b however one sees that even the small noise level corresponding to  $\mathcal{R}_c = 60 \text{ db}$  leads to a large displacement of the optimum pass-band down to about  $2.7W_q$  cps. For larger noise levels the optimum frequency response function very rapidly

approaches the matched frequency response function (essentially identical to the curves for  $\mathcal{R}_c = 0$  db in Figs. 11 and 12).

The cases depicted in Figs. 11c and 12c show similarly great differences for  $\mathcal{R}_c = \infty$  and  $\mathcal{R}_c = 60$  db. In all cases the primary effect is the reduction to (essentially) zero of the large values of the frequency response function which can exist for large values of  $f/W_q$  when  $N_o = 0$ . These reductions have their common origin in the fact that for such values of  $f$  the numerator of Eq. (10.50) is small compared to the  $N_o = 0.0001$  which constitutes essentially the entire denominator value at the same frequencies.†

Thus the marked differences in performance originally noted in Figs. 10a and 10b are seen to be rooted in correspondingly great differences in the optimum frequency response functions for zero-noise and small-noise situations. Figures 11 and 12 therefore reinforce the primary conclusion that the small-noise solution cannot necessarily be regarded as a small deviation from the no-noise solution.

#### D. PERFORMANCE IMPROVEMENT OVER THE SIMPLE MATCHED FILTER

The performance differences which have been seen to exist between the optimum and matched processors considered in this chapter may be compared to the general performance improvement bound derived in chapter nine.

It is convenient to continue to use of decibel measure for  $\frac{S}{I}$  and to define the performance difference  $\Delta(\mathcal{R}_c, f_o, W)$  by

$$\Delta(\mathcal{R}_c, f_o, W) = 10 \log \left( \frac{S}{I} \right)_{\text{opt}} - 10 \log \left( \frac{S}{I} \right)_{\text{mf}} \quad (10.58)$$

† See Appendix E for other parameter values.



where the fact that  $\Delta$  depends upon each of (i) clutter-to-noise ratio  $R_c$ , (ii) echo doppler frequency  $f_o$ , and (iii) echo bandwidth  $W$ , has been explicitly indicated in the notation. Actually  $\Delta$  also depends upon the clutter bandwidth  $W_q$ . Since the dependence, however, is only through the ratios  $f_o/W_q$  and  $W/W_q$ , the simplified notation adopted in eq. (10.58) is adequate for the present discussion.<sup>†</sup>

The difference  $\Delta(R_c, f_o, W)$ , a function of three variables, is now to be compared to the bound  $B(R_c)$ , which is a function of only one variable. Since, however, the present discussion is primarily concerned with the maximum value which  $\Delta(R_c, f_o, W)$  might achieve, the task of comparison is simplified by considering the function  $\Delta_W$  defined by

$$\Delta_W(R_c, f_o) = \max_W \Delta(R_c, f_o, W) \quad (10.59)$$

Values of the function  $\Delta_W(R_c, f_o)$  can be determined from the data already graphed in Figs. 10b through 10h. The procedure is suggested by Fig. 13, where the curves for  $(\frac{S}{I})_{opt}$  and  $(\frac{S}{I})_{mf}$  corresponding to  $R_c = 60$  db and  $f_o = 2W_q$  have been reproduced from Fig. 10b. For these values of  $R_c$  and  $f_o$ , the value of  $\Delta_W(R_c, f_o)$  is then given by the greatest vertical separation of the two performance curves. As shown, the value of  $\Delta_W(60 \text{ db}, 2W_q)$  is about 24 db.

By using other pairs of curves from Fig. 10, one may determine other values of  $\Delta_W(R_c, f_o)$ . Figure 14 contains

---

<sup>†</sup> One may consider  $W_q$  to be fixed throughout the discussion.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

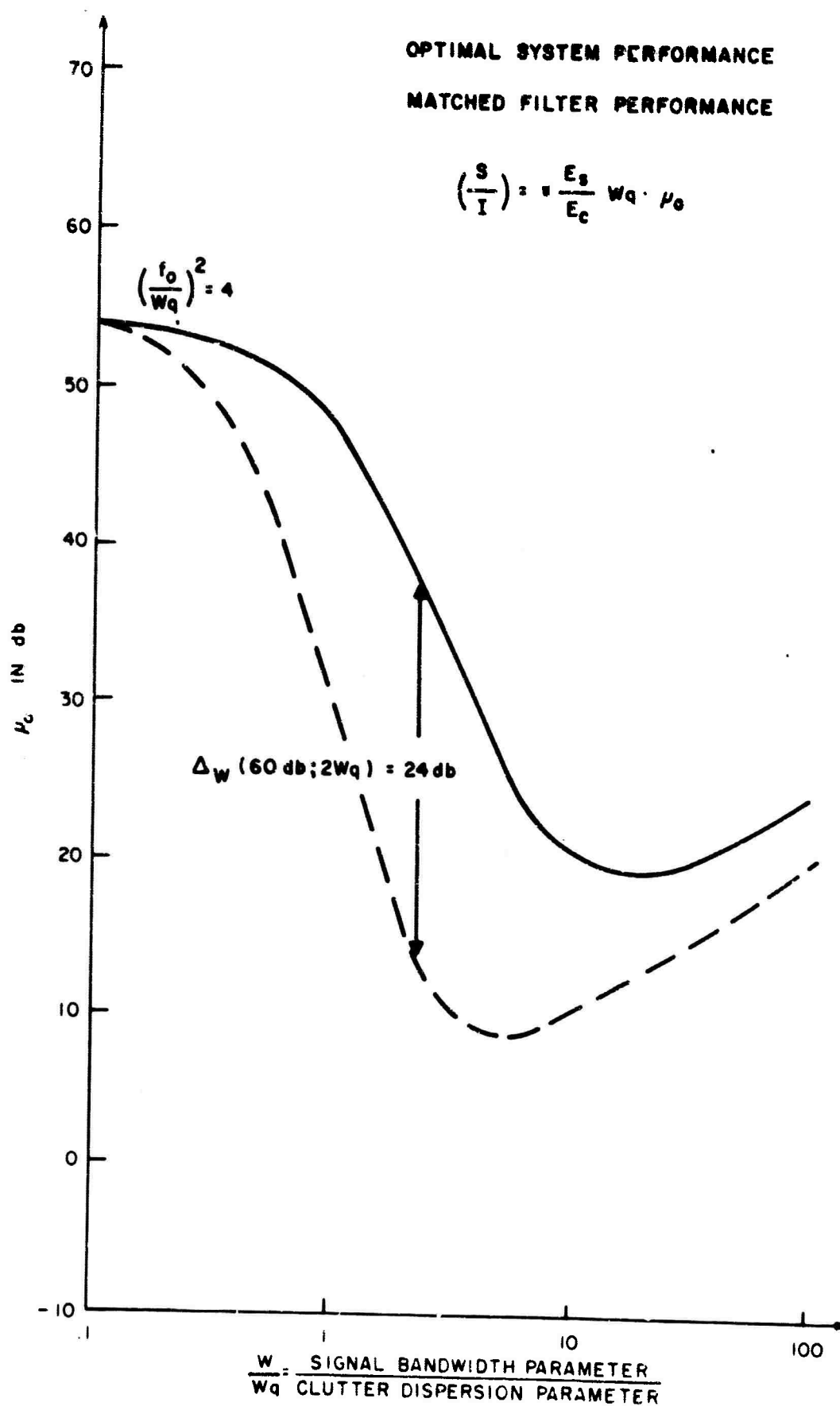


FIG. 13 ILLUSTRATIVE EXAMPLE OF MAXIMUM PERFORMANCE DIFFERENCE

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## NOTES:

- 1) DATA IS FOR GAUSSIAN PULSE IN UNIFORMLY EXTENDED GAUSSIAN CLUTTER.
- 2) ECHO DOPPLER  $f_0 = 2Wq$

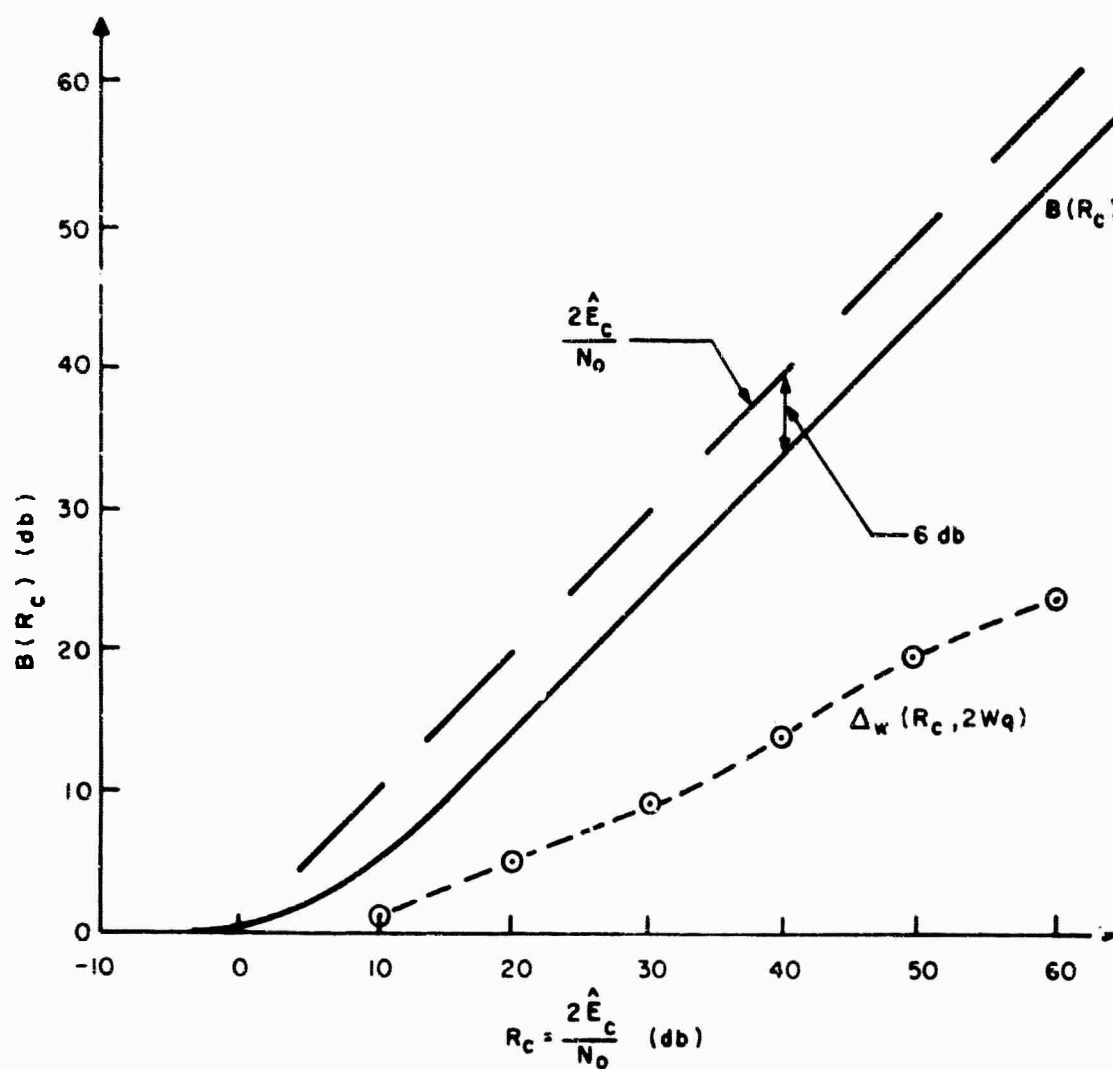


FIG. 14 EMPIRICAL PERFORMANCE IMPROVEMENT DATA AS A FUNCTION OF CLUTTER-TO-NOISE RATIO

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

the seven values for

$$\mathcal{R}_c = 0, 10, 20, 30, 40, 50, 60 \text{ db} \quad (10.60a)$$

and

$$f_o = 2W_q. \quad (10.60b)$$

From the data of Fig. 14 one notices that

- (i) the indicated function values for  $\Delta_w(\mathcal{R}_c, f_o)$  lie below  $B(\mathcal{R}_c)$ , as required;
- (ii) the function  $\Delta_w(\mathcal{R}_c, f_o)$  is an increasing function of  $\mathcal{R}_c$ , at least for  $f_o = 2W_q$ , just as the bound  $B(\mathcal{R}_c)$  itself is;
- (iii) the difference between  $B(\mathcal{R}_c)$  and  $\Delta_w(\mathcal{R}_c, 2W_q)$  increases with  $\mathcal{R}_c$  until, at  $\mathcal{R}_c = 60 \text{ db}$ ,  $\Delta_w(\mathcal{R}_c, 2W_q)$  falls short of  $B(\mathcal{R}_c)$  by about 30 db.

These data clearly indicate that, at least for the present case with echo doppler equal to  $2W_q$ , the potentially large performance improvements "permitted" by large values of the bound  $B(\mathcal{R}_c)$  may exceed by many decibels the actual performance improvements which may be achieved.

The natural question arises concerning how much the actual performance improvements shown in Fig. 14, for one particular echo doppler frequency, might be increased for echoes with other values of  $f_o$ .

An indication of the dependence of performance improvement upon doppler frequency may be had by consideration of Figs. 15 and 16.

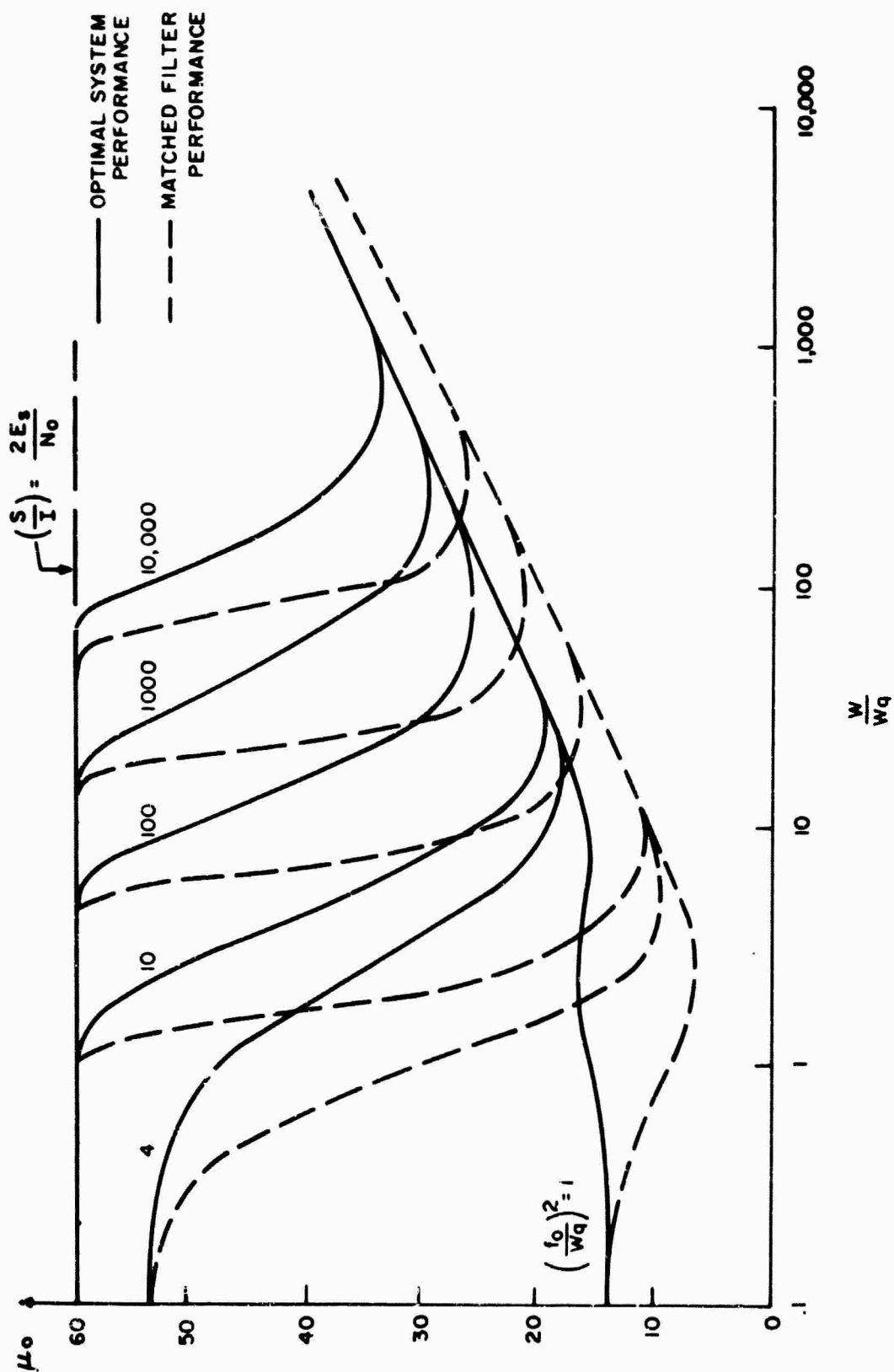


FIG. 15 OPTIMUM AND MATCHED PROCESSOR PERFORMANCE FOR VARIOUS DOPPLER SHIFTS, AND  $R_c = 60\text{db}$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## NOTES:

- 1) DATA IS FOR GAUSSIAN PULSE IN UNIFORMLY EXTENDED GAUSSIAN CLUTTER.
- 2) ECHO DOPPLER  $f_0 = 2 W_q$
- 3) CLUTTER-TO-NOISE RATIO  $R_c = \frac{2\hat{E}_c}{N_0} = 60\text{db}$

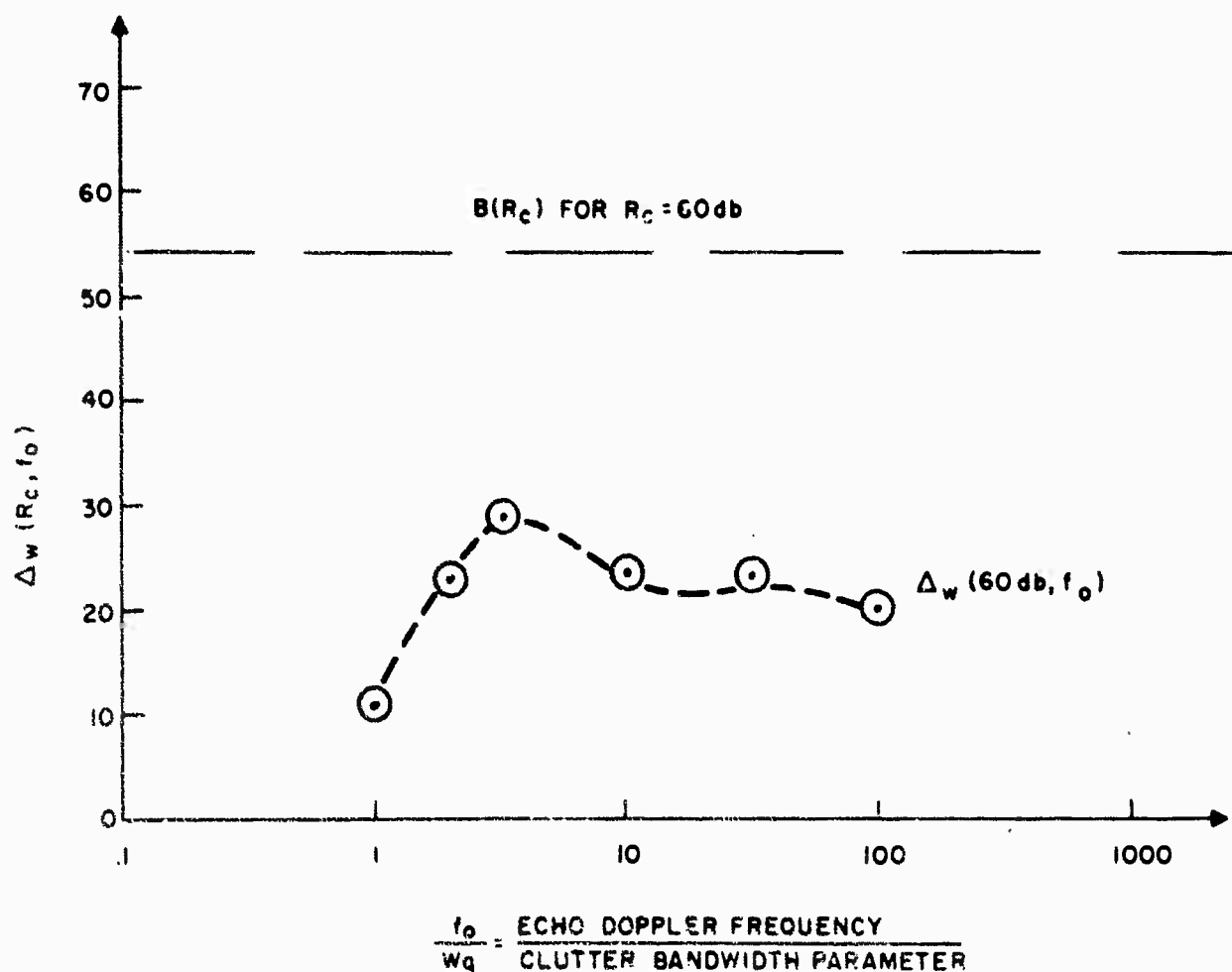


FIG. 16 EMPIRICAL PERFORMANCE IMPROVEMENT DATA AS A FUNCTION OF ECHO DOPPLER FREQUENCY

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

In Fig. 15, optimum and matched processor performance data are plotted as a function of echo bandwidth  $W$ , for several values of echo doppler frequency  $f_o$  and the single value of  $R_c$  equal to 60 db. The greatest difference between each pair of curves yields, in the manner of Fig. 13, a single point shown in Fig. 16. By this procedure one develops a profile of values of  $\Delta_w(R_c, f_o)$  considered as a function of  $f_o$  with  $R_c$  fixed.

One sees in Fig. 15 the very pronounced tendency, already noted, of system performance for both optimum and matched processors to improve with increasing doppler frequency separation between the echo and the bulk of the clutter energy. Also quite evident is the termination of this general increase of signal-to-interference ratio when the maximum value of  $2\mathcal{E}_s/N_o$  is attained (see eq. 10.56).

In Fig. 16 one sees that the actual performance difference  $\Delta_w(R_c, f_o)$  between optimum and matched processors tends at first to increase with increasing doppler frequency shift  $f_o$ . For some  $f_o$  in the vicinity of about  $5W_q$ , however, the data suggests that  $\Delta_w(R_c, f_o)$  reaches a maximum and then begins to decline for larger  $f_o$ .

Thus Fig. 16 suggests an improvement of perhaps 5 db over the data of Fig. 14 for  $R_c$  equal to 60 db. More specifically, Fig. 16 indicates that, for  $R_c = 60$  db, the greatest performance difference which can exist between optimum and matched processors is about 30 db. This difference occurs (at one particular echo bandwidth) for an echo doppler frequency in the vicinity of  $5W_q$  cps. It is about 25 db below the performance improvement bound  $B(R_c)$ .

XI. DETECTION OF GAUSSIAN PULSES IN CLUTTER  
WITH GAUSSIAN DELAY AND DOPPLER PROFILES

In the example of the preceding chapter, the clutter energy was assumed to be uniformly distributed over all values of range delay. In practice, however, a clutter source will certainly not have the infinite physical dimension implied by such an assumption. Rather, the clutter source might be restricted to only certain range delays of interest, with the result that some signal echoes might be received in noise only, while other echoes were heavily masked by clutter. The effect upon the optimum processor of such spatial (or range delayed) variations of clutter energy is a question of some interest. In this chapter a clutter dispersion function is considered which has significant variation over both range delay and doppler frequency shift.

A. ANALYTIC RESULTS

The clutter dispersion function to be studied is given by

$$\mathcal{E}(\rho, f) = \mathcal{E}_c \cdot \mathcal{E}(\rho) \cdot Q(f) \quad (11.1)$$

where  $\mathcal{E}(\rho) = \exp \left\{ - \pi \frac{\rho^2}{D^2} \right\}$  (11.2)

and  $Q(f) = \frac{1}{W_q} \cdot \exp \left\{ - \pi \frac{f^2}{W_q^2} \right\}$  (11.3)

The doppler profile,  $Q(f)$ , is that of the preceding section (see Eq. 10.8), while the delay profile is chosen to yield maximum energy at delay  $\rho$  equal to zero. This is a conven-



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

ient choice for the origin relative to which echo delays will be specified.

The transmitter modulation is again taken to be

$$m(t) = 2^{\frac{1}{4}} W^{\frac{1}{2}} \cdot \exp \left\{ - \pi W^2 t^2 \right\} \quad (11.4)$$

as in the preceding chapter.

Under these circumstances it may be verified that Eq. (4.18) for the clutter covariance function yields

$$\mathcal{K}_c(t_1, t_2) = \mathcal{E}_c \cdot g(t_1) g(t_2) \cdot k(t_1 - t_2) \mathcal{Q}(t_1 - t_2) \quad (11.5)$$

$$\text{where } g(t) = \frac{2^{\frac{1}{4}} W^{\frac{1}{2}} D^{\frac{1}{2}}}{(2W^2 D^2 + 1)^{\frac{1}{4}}} \cdot \exp \left\{ - \pi \frac{W^2 t^2}{2W^2 D^2 + 1} \right\} \quad (11.6)$$

$$k(t_1 - t_2) = \exp \left\{ - \pi \frac{W^4 D^2 (t_1 - t_2)^2}{2W^2 D^2 + 1} \right\} \quad (11.7)$$

$$\text{and } \mathcal{Q}(t_1 - t_2) = \exp \left\{ - \pi W_q^2 (t_1 - t_2)^2 \right\}. \quad (11.8)$$

It will be noted that as  $D \rightarrow \infty$  in Eq. (11.2), the function  $\mathcal{G}(\rho) \rightarrow 1$  for all  $\rho$ , and the results of this chapter should approach those in the preceding chapter for uniformly extended clutter. This indeed happens for, as  $D \rightarrow \infty$ ,

$$g(t) \rightarrow 1 \text{ for all } t \quad (11.9)$$

$$\text{and } k(t_1 - t_2) \rightarrow \exp \left\{ - \pi \frac{W^2}{2} (t_1 - t_2)^2 \right\} \quad (11.10)$$

When Eqs. (11.5), (11.9), and (11.10) are compared to Eq. (10.6) it is seen that  $\mathcal{K}_c(t_1, t_2)$  for the present case does approach  $\mathcal{K}_c(t_1, t_2)$  for the case of uniformly extended clutter.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The fact that  $\mathcal{N}_c(t_1, t_2)$  in the present case is not a function of the difference  $(t_1 - t_2)$ , however, necessitates a different approach to the solution.

If the explicit form for  $\mathcal{N}_c(t_1, t_2)$  given by Eq. (11.5) is inserted into the general expression (4.33) for signal-to-interference ratio, the result is

$$\frac{S}{I} = \frac{2\mathcal{E}_s \left| \int_{-\infty}^{\infty} w^*(t; \rho_o, f_o) m(t; \rho_o, f_o) dt \right|^2}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1; \rho_o, f_o) g(t_1) \cdot \mathcal{E}_c k(t_1 - t_2) \mathcal{Q}(t_1 - t_2) \cdot g(t_2) w(t_2; \rho_o, f_o) dt_1 dt_2} \quad (11.11)$$

where  $\frac{S}{I}$  is to be maximized by appropriate choice of  $w(t; \rho_o, f_o)$ . Equation (11.11), however, suggests the substitution

$$y^*(t; \rho_o, f_o) = g(t) w^*(t; \rho_o, f_o), \quad (11.12)$$

in order to reduce  $\frac{S}{I}$  to

$$\frac{S}{I} = \frac{2\mathcal{E}_s \left| \int_{-\infty}^{\infty} y^*(t; \rho_o, f_o) \cdot \frac{m(t; \rho_o, f_o)}{g(t)} \cdot dt \right|^2}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^*(t_1; \rho_o, f_o) \cdot \mathcal{E}_c k(t_1 - t_2) \mathcal{Q}(t_1 - t_2) \cdot y(t_2; \rho_o, f_o) dt_1 dt_2} \quad (11.13)$$

That this expression for  $\frac{S}{I}$  has exactly the form of Eq. (4.48) may now be verified.

Equations (4.50) through (4.54) may therefore be used to write the solution for that  $y(t; \rho_o, f_o)$  which maximizes the ratio given by Eq. (11.13). The result is

$$\mathcal{F}\{y(t; \rho_o, f_o)\} = \frac{\left[ \mathcal{F}\left\{ \frac{m(t; \rho_o, f_o)}{g(t)} \right\} \right]}{\mathcal{F}\{\mathcal{E}_c \cdot k(t_1 - t_2) \mathcal{Q}(t_1 - t_2)\}} \quad (11.14)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

from which  $w_{opt}(t; \rho_o, f_o)$  may be found by using Eq. (11.12).  
Thus

$$w_{opt}(t; \rho_o, f_o) = \frac{1}{g(t)} \cdot \mathcal{F}^{-1} \left\{ \mathcal{F}\{y(t; \rho_o, f_o)\} \right\} \quad (11.15)$$

on the condition that the transforms indicated in both Eqs. (11.14) and (11.15) exist.

Before proceeding further it must be noted that Eq. (11.14) for  $y(t; \rho_o, f_o)$  gives the maximum value to  $\frac{S}{I}$ , subject to the constraint†

$$\left| \int_{-\infty}^{\infty} y^*(t; \rho_o, f_o) \cdot \frac{m(t; \rho_o, f_o)}{g^*(t)} \cdot dt \right|^2 = \text{const.} = |K|^2 \quad (11.16)$$

By virtue of Eq. (11.12), however, this is identically the constraint

$$\left| \int_{-\infty}^{\infty} w^*(t; \rho_o, f_o) m(t; \rho_o, f_o) dt \right|^2 = |K|^2 \quad (11.17)$$

which is desired in finding the  $w(t; \rho_o, f_o)$  to maximize  $\frac{S}{I}$  given by Eq. (11.11).

The process is therefore justified, of solving the constrained extremum problem of Eq. (11.13), with constraint Eq. (11.16), and then finding  $w_{opt}(t; \rho_o, f_o)$  via Eq. (11.12). The same result must ensue as if the original extremum problem of Eq. (11.11) were solved directly for  $w_{opt}(t; \rho_o, f_o)$ , subject to the constraint Eq. (11.17)

With the validity of the solution (11.15) established, the next concern is an expression for the maximum value of  $\frac{S}{I}$ . Equation (4.51) may be used to write

---

† See Eqs. (A-1) and (A-2) for the origins of the constraint.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$\left(\frac{S}{I}\right)_{\text{opt}} = \frac{2\mathcal{E}_s}{\mathcal{E}_c} \int_{-\infty}^{\infty} \frac{\left| \mathcal{F}\left\{ \frac{m(t; \rho_o, f_o)}{q^*(t)} \right\} \right|^2}{\mathcal{F}\{k(t_1 - t_2) \mathcal{Q}(t_1 - t_2)\}} df \quad (11.18)$$

which employ the functions appearing in the (simpler) alternative Eq. (11.13).

When the particular Eqs. (11.4) through (11.8) for the present case are used in the more general expressions (11.14) and (11.18), the results are

$$y(t; \rho_o, f_o) \propto \exp \left\{ -\pi \left( t - \rho_o \cdot \frac{w^2}{w_k^2 - w_q^2} \right)^2 \left( \frac{w_k^2 (w^2 - w_q^2)}{w_k^2 - (w^2 - w_q^2)} \right) \right\} \\ \cdot \exp \left\{ j2\pi f_o \cdot \left( \frac{w_k^2}{w_k^2 - (w^2 - w_q^2)} \right) \right\} \quad (11.19)$$

and

$$\left(\frac{S}{I}\right)_{\text{opt}} = \frac{\mathcal{E}_s w_q}{\mathcal{E}_c} \cdot \left[ 1 + \frac{1}{2w^2 D^2} + \frac{w^2}{2w_q^2} \right] \cdot \exp \left\{ \pi \frac{\rho_o^2}{D^2} \right\} \cdot \exp \left\{ \pi \frac{f_o^2}{w_q^2} \right\} \quad (11.20)$$

where

$$w_k^2 = w_q^2 + \frac{w^2}{2} \cdot \left( \frac{2w^2 D^2}{2w^2 D^2 + 1} \right) \quad (11.21)$$

An expression for  $w_{\text{opt}}(t; \rho_o, f_o)$  can be derived from Eqs. (11.19), (11.15), and (11.6). However, a more informed understanding of the functioning of the optimum processor can be achieved from the viewpoint to be presented in the next section.

## B. DISCUSSION OF ANALYTIC RESULTS

For the particular assumptions of a "bi-Gaussian" dispersion function, and a Gaussian pulse echo, the clutter covariance function has been seen to be factorable in the form of

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Eq. (11.5). It is readily verified, therefore, that the mean-square clutter interference at any time  $t$  is given by

$$\mathcal{H}_c(t, t) = \mathcal{E}_c \cdot g^2(t) \quad (11.22)$$

The mean clutter level is therefore variable with time, and the clutter process itself is statistically non-stationary. The particular non-stationarity which is implied by Eq. (11.5) for  $\mathcal{H}_c(t_1, t_2)$ , however, is of a rather "simple" kind.

A process  $x(t)$  with a covariance function  $\mathcal{H}_x(t_1, t_2)$  which is factorable in the form

$$\mathcal{H}_x(t_1, t_2) = a(t_1)K(t_1 - t_2)a^*(t_2) \quad (11.23)$$

may be regarded as having been derived from a stationary process  $y(t)$ , with covariance function

$$\mathcal{H}_y(t_1, t_2) = K(t_1 - t_2), \quad (11.24)$$

by passing  $y(t)$  through an instantaneous amplifier with amplification varying as  $a(t)$ . Under these circumstances one will have

$$x(t) = a(t)y(t) \quad (11.25)$$

which yields

$$\begin{aligned} \mathcal{H}_x(t_1, t_2) &= \langle x(t_1)x^*(t_2) \rangle \\ &= \langle a(t_1)y(t_1)y^*(t_2)a^*(t_2) \rangle \\ &= a(t_1) \langle y(t_1)y^*(t_2) \rangle a^*(t_2) \\ &= a(t_1)K(t_1 - t_2)a^*(t_2) \end{aligned} \quad (11.26)$$

as required by Eq. (11.23).

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Although this is not the physical mechanism through which the particular clutter interference of the present case arises, it is a convenient mathematical framework for describing the operation and structure of the optimum (non-stationary) processor.

### 1. Structure of the Optimum Processor

At the left of Fig. 17 is shown an equivalent source for the clutter interference being considered. A stationary random process  $x(t)$  is assumed to exist and to pass through an instantaneous, time-variable amplifier. The amplification is assumed to vary as  $g(t)$ . The covariance function of the process  $x(t)$  is taken to be  $\mathcal{E}_c \cdot k(\tau) \cdot \mathcal{Q}(\tau)$ , as indicated in the figure.

The result is that the process emerging from the equivalent source has the desired covariance function, namely

$$\mathcal{K}_c(t_1, t_2) = \mathcal{E}_c g(t_1) k(t_1 - t_2) \mathcal{Q}(t_1 - t_2) g(t_2) \quad (11.27)$$

as indicated in Fig. 17. As the optimum processor requires no more knowledge of the interference than its covariance function, the interference assumed to be generated in the manner of Fig. 17 is an acceptable substitute for the actual clutter process.

To the (equivalent) clutter interference is added the signal echo  $m(t; \rho_0, f_0)$  to be detected. The sum of echo and clutter is then available at the processor input. In general, noise would also be part of the sum, as indicated by the dotted line in Fig. 17. In the present analysis, however, noise will be neglected.

For this situation, the optimum processor derived algebraically in the preceding section may be regarded as the tandem

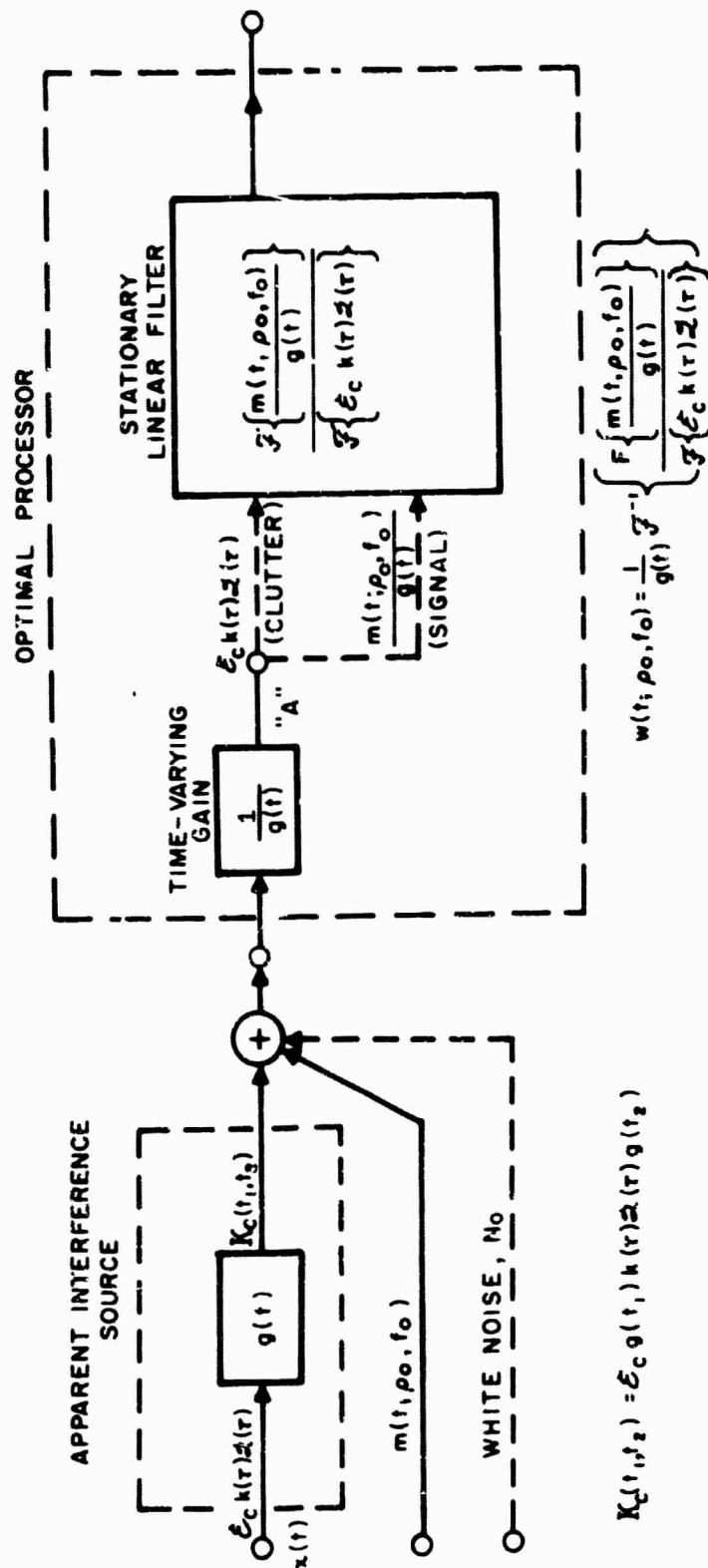


FIG. 17 OPTIMUM PROCESSOR FOR A PARTICULAR SORT OF NON-STATIONARY INTERFERENCE ( $N_0 = 0$ )

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

combination of a time-variable amplifier and a stationary, linear filter. The structure implied by Eq. (11.15) is shown at the right in Fig. 17. The amplification varying as  $1/g(t)$  corresponds to the leading term on the right of Eq. (11.15), while the stationary filter of Fig. 17 is assumed to generate the time function  $\mathcal{F}^{-1} \{ \mathcal{F}\{y(t; \rho_o, f_o)\} \}$  which appears in Eq. (11.15)

The two operations performed within the optimum processor may now be considered separately

The action of the variable amplifier in the optimum processor upon the clutter component of the input waveform is clearly to undo the previous multiplication by  $g(t)$  in the apparent interference source. At the point "A" within the optimum processor, therefore, the clutter has been converted to a stationary process with covariance function given by the original

$$\mathcal{K}_x(t_1, t_2) = \mathcal{E}_c \cdot k(t_1 - t_2) \mathcal{Q}(t_1 - t_2). \quad (11.28)$$

The signal echo which enters the optimal processor as  $m(t; \rho_o, f_o)$  appears at point "A" as the waveform

$$z(t; \rho_o, f_o) = \frac{m(t; \rho_o, f_o)}{g(t)}. \quad (11.29)$$

Now the optimum processor  $y(t; \rho_o, f_o)$  for detecting the signal  $z(t; \rho_o, f_o)$  in the stationary interference  $x(t)$  has already been defined by Eq. (4.50). The optimum processor is therefore given by



$$\mathcal{F}\{y(t; \rho_o, f_o)\} = \frac{\left[ \mathcal{F}\left\{ \frac{m(t; \rho_o, f_o)}{g(t)} \right\} \right]^*}{\mathcal{F}\{ \epsilon_c k(\tau) \mathcal{Q}(\tau) \}} \quad (11.30)$$

where equations (11.28) and (11.29) have provided the necessary terms for use with equation (4.50).

The processor  $y(t; \rho_o, f_o)$  derived in this manner and given by equation (11.30) is observed to be identical to the processor  $y(t; \rho_o, f_o)$  derived algebraically, and given by equation (11.14). It is the processor shown as the second component of the optimal processor  $w_{opt}(t; \rho_o, f_o)$  in Figure 17 .

From the viewpoint of Figure 17 , and the recent discussion, optimum processor operation in the present case consists of two actions:

- i) First the incoming waveform has its amplitude appropriately varied as a function of time, to render the interfering process statistically stationary.
- ii) Then the optimum processor for detection in the resulting stationary interference is found.

The tandem combination of both actions yields the optimum (non-stationary) processor for the original problem.

This manner of interpreting optimum processor operation is possible whenever the interference covariance function has the particular form of equation (11.5).

Note that the inclusion of noise in this case would cause the interference covariance function not to be factorable

in the necessary form. Neither the algebraic nor the intuitive mode of solution can therefore be carried out if the presence of noise is assumed.

## 2. Signal-to-Interference Ratio

The signal-to-interference ratio for this optimum processor has been given in equation (11.20). This equation, in fact, is seen to be remarkably similar to the expression (10.18) for  $\frac{S}{I}$  in the case of uniformly extended Gaussian clutter. Indeed equation (11.20) reduces as it should, to equation (10.18) as the clutter extent parameter  $D$  approaches infinity.

For any finite extent parameter  $D$ , it may be verified that  $\frac{S}{I}$  of equation (11.20) exceeds  $\frac{S}{I}$  of equation (10.18). One might say that the improved performance in localized clutter was due to the added possibility of discriminating against the clutter energy on the basis of range delay  $\rho_0$ , in addition to the discrimination on the basis of frequency spectra as described in chapter ten. Such an interpretation does accord with one's intuitive expectation, based on Figure 18, for this case of non-uniform clutter. It is supported by the observation, from equation (11.20), that performance does improve as the range delay,  $\rho_0$ , between the maximum clutter energy (at  $\rho = 0$ ) and the maximum echo energy (at  $\rho = \rho_0$ ) increases.

The effects of noise upon the optimum solution for the present case may be expected to be quite similar to the effects described in chapter ten for the case of uniformly extended clutter. The two zero-noise solutions have many points of similarity, with the result that much of the discussion of chapter ten has analogous application in the present case.

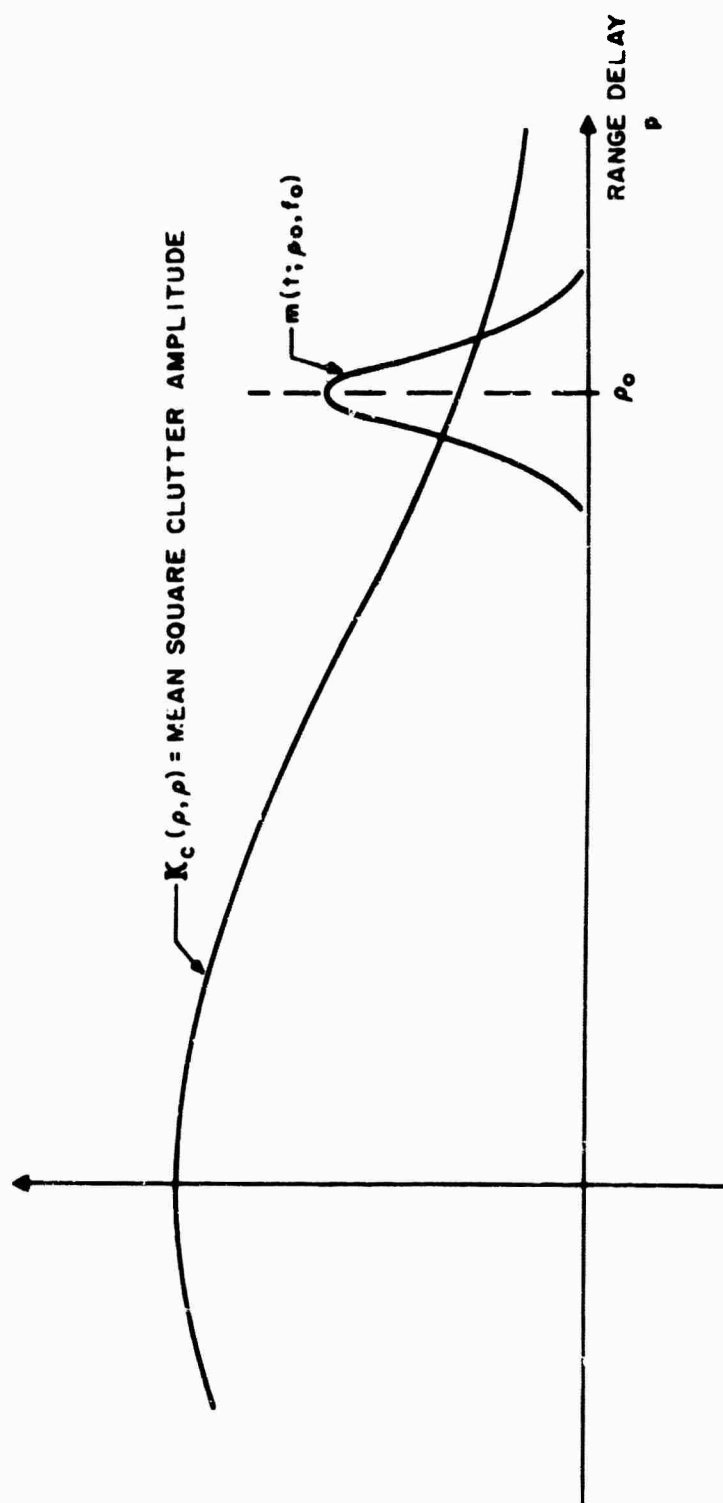


FIG. 18 RANGE DELAY PROFILE OF GAUSSIAN PULSE IN NON -  
STATIONARY GAUSSIAN CLUTTER

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The first point of similarity exists in the stationary component of the optimum processor shown in Figure 17. The Fourier transform of its weight function is given by equation (11.30) as a ratio with denominator equal to  $\mathcal{F}\{e_c k(\tau) \mathcal{Q}(\tau)\}$ . The function  $k(\tau) \mathcal{Q}(\tau)$  is Gaussian in the present case (cf. equations 11.7 and 11.8) with the result that  $\mathcal{F}\{e_c k(\tau) \mathcal{Q}(\tau)\}$  is also Gaussian and rapidly approaches zero as the frequency variable of the transform approaches infinity. The result is that the weight function  $y(t; \rho_0, f_0)$  of the stationary component of the optimum processor for the present case will tend to emphasize higher frequency components of the signal at its input, with essentially similar results to those shown and discussed earlier in connection with Figures 6 and 7. In the previous case the high-frequency emphasis arose out of the analogous presence of  $\mathcal{F}\{m(\tau) \mathcal{Q}(\tau)\}$  in the denominator of the optimum processor (see equations 4.45 and 10.20).

The optimum processor for the present case possesses a second point where the presence of noise will greatly modify processor operation. The preliminary time-variable amplification by  $1/g(t)$  shown in Figure 17 provides minimum amplification at  $t = 0$ , but rapidly increasing amplification for  $t \rightarrow \pm \infty$ . The reason is again the presence of a Gaussian function in the denominator of a ratio. Here it is  $g(t)$  which tends rapidly to zero for large  $t$ . Again the presence of noise will militate against the unbounded amplifications which are indicated for the noise-free case. It is to be expected that if a time-variable amplifier is a component of the optimum processor for bi-Gaussian clutter plus noise, then its amplification may well be minimum where clutter is large, but it will almost certainly be constant (rather than unboundedly increasing) where clutter is small.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

An exact description of the effects of noise in the present case is unfortunately not possible, because of the form of the solution. In the preceding case, equation (10.17) gave the performance essentially in terms of an eigenfunction expansion of the problem solution<sup>†</sup>, and noise was simply included by appropriate modification of eigenvalues. In the present case, however, equations (11.15) and (11.18) for the solution are not given in terms of eigenfunctions of the covariance kernel. The result is that the effects of noise cannot be directly incorporated into the no-noise solution.

This solution for bi-Gaussian clutter therefore provides an example where even complete knowledge of the zero-noise solution is not necessarily of help in finding the optimum solution for interference which includes low noise levels.

---

<sup>†</sup> Note the formal similarity of equations (10.17) and (6.33).

## XII. DETECTION OF RECTANGULAR PULSES IN UNIFORMLY EXTENDED CLUTTER FROM A STATIONARY SOURCE

This chapter and the next will deal with clutter arising from a source which introduces no Doppler dispersion among the clutter components. Physically, this would correspond to reflection of the transmitted waveform from a spatial distribution of essentially motionless scattering centers.

The simpler cases, which arise when the mean clutter energy is uniform for all range delays, are considered in this chapter. The results to be given, in addition to having intrinsic interest, serve also as the basis for extensions to the more general cases in chapter XIII.

### A. THE DIFFERENCE-DIFFERENTIAL EQUATION

The clutter dispersion function  $\mathcal{E}(\rho, f)$  is assumed to have the form

$$\mathcal{E}(\rho, f) = \mathcal{E}_c \cdot \delta(f) \quad (12.1)$$

for all range delay  $\rho$ . Introduction of this expression into the general formula (4.16) for the clutter covariance kernel then leads to

$$\mathcal{K}_c(t_1, t_2) = 2 \cdot \int_{-\infty}^{\infty} \mathcal{E}_c \cdot m(t_1; \rho) m^*(t_2; \rho) d\rho, \quad (12.2)$$

where

$$m(t_1; \rho) \equiv m(t_1; \rho, 0) \equiv m(t_1 - \rho). \quad (12.3)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

Equation (12.2) will be observed to have the same form as Eq. (8.28).

The modulation envelope  $m(t)$  is assumed to be rectangular with unit integral-square-amplitude, as required by Eq. (4.21). The envelope

$$m(t) = \begin{cases} T^{-\frac{1}{2}} & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \quad (12.4)$$

corresponds to a single pulse of duration  $T$  seconds - the elementary radar signal. It is clear that for this pulse

$$\frac{d}{dt} m(t) = \frac{1}{T^{\frac{1}{2}}} \delta(t + \frac{T}{2}) - \frac{1}{T^{\frac{1}{2}}} \delta(t - \frac{T}{2}) \quad (12.5)$$

which is precisely the form of Eq. (8.30), namely

$$L_t m(t) = M_t D_t \delta(t) , \quad (12.6)$$

where

$$L_t f \equiv \frac{d}{dt} f(t) \quad (12.7a)$$

$$M_t f \equiv T^{-\frac{1}{2}} \cdot f(t) \quad (12.7b)$$

and

$$D_t f \equiv f(t + \frac{T}{2}) - f(t - \frac{T}{2}) . \quad (12.7c)$$

It may be verified<sup>†</sup> that the adjoint to  $L_t$  is given by

$$\tilde{L}_t f \equiv - \frac{d}{dt} f(t) . \quad (12.7d)$$

---

<sup>†</sup> See Eq. (8.20)

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

With only real constants involved in these definitions, it follows, moreover, that complex conjugation of an operator leaves the operator unchanged.

When the operators defined in the preceding four equations are applied in Eqs. (8.45) and (8.46), the respective results are

$$w(t; \rho_o, f_o) = - \frac{d}{dt} \psi^*(t; \rho_o, f_o) \quad (12.8a)$$

and

$$\frac{2\hat{\epsilon}}{T} \cdot D_t D_t^+ \psi^*(t; \rho_o, f_o) - N_o \cdot \frac{d^2}{dt^2} \psi^*(t; \rho_o, f_o) = \frac{d}{dt} m(t; \rho_o, f_o) \quad (12.8b)$$

This system of equations may be reduced to a single equation by differentiating the second with respect to  $t$  and then using Eq. (12.8a). The result is

$$- \frac{2\hat{\epsilon}}{T} D_t D_t^+ w(t; \rho_o, f_o) + N_o \frac{d^2}{dt^2} w(t; \rho_o, f_o) = \frac{d^2}{dt^2} m(t; \rho_o, f_o) \quad (12.9)$$

or if the extended form, Eq. (12.7c), is used for  $D_t$ ,<sup>†</sup>

$$\begin{aligned} \frac{2\hat{\epsilon}}{T} \left\{ w(t+T; \rho_o, f_o) - 2 \cdot w(t; \rho_o, f_o) + w(t-T; \rho_o, f_o) \right\} \\ + N_o \cdot \frac{d^2}{dt^2} w(t; \rho_o, f_o) = \frac{d^2}{dt^2} m(t; \rho_o, f_o) . \end{aligned} \quad (12.10)$$

<sup>†</sup> It is a quirk of this particular operator  $D$ , that

$$-D_t D_t^+ \equiv D_t D_t$$



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The problem is now to solve Eq. (12.10) for  $w(t; \rho_o, f_o)$ . In words, Eq. (12.10) asserts that the unknown weight function is characterized by a weighted sum of its second derivative and second difference being equal to the second derivative of the desired echo. Visualization of the unknown function at this point is not necessarily an easy matter. The simplest of radar waveforms together with the simplest clutter model has, unfortunately, not led to the simplest of equations.

## B. A GENERAL SOLUTION

Equation (12.10) may be solved by assuming that both  $w(t; \rho_o, f_o)$  and  $m(t; \rho_o, f_o)$  admit representations as Laplace integrals in the following manner:

$$w(t; \rho_o, f_o) = \int_C W(s; \rho_o, f_o) e^{st} ds \quad (12.11a)$$

$$m(t; \rho_o, f_o) = \int_C M(s; \rho_o, f_o) e^{st} ds \quad (12.11b)$$

where  $s = \sigma + j\omega$ , and  $C$  is the contour extending from  $s = -j\infty$  to  $s = +j\infty$  along the path  $\sigma = 0$ . The direct bilateral transforms are given by

$$W(s; \rho_o, f_o) = \int_{-\infty}^{\infty} w(t; \rho_o, f_o) e^{-st} dt \quad (12.12a)$$

and

$$M(s; \rho_o, f_o) = \int_{-\infty}^{\infty} m(t; \rho_o, f_o) e^{-st} dt \quad (12.12b)$$

When the representations (12.11) are introduced into Eq. (12.10) there results, in exactly analogous fashion to solution by ordinary, one-sided Laplace transforms,

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$\left[ N_o \cdot s^2 + \frac{2 \mathcal{E}_c}{T} \left\{ e^{+sT} - 2 + e^{-sT} \right\} \right] W(s; \rho_o, f_o) = s^2 \cdot M(s; \rho_o, f_o) \quad (12.13)$$

Algebraic manipulation followed by solution for  $W(s; \rho_o, f_o)$  yields

$$W(s; \rho_o, f_o) = \frac{s^2}{N_o \cdot s^2 + \frac{2 \mathcal{E}_c}{T} \cdot 4 \sinh^2\left(\frac{sT}{2}\right)} \cdot M(s; \rho_o, f_o) \quad (12.14)$$

where, using Eqs. (12.4) and (12.12b), it may be verified that

$$M(s; \rho_o, f_o) = \frac{T^{-\frac{1}{2}}}{s - j2\pi f_o} \cdot \sinh\left\{ \frac{(s - j2\pi f_o)T}{2} \right\} \cdot e^{-(s - j2\pi f_o)\rho_o} \quad (12.15)$$

In principle one may now use Eq. (12.14) in the integral (12.11a) to accomplish the necessary inversion for  $w(t; \rho_o, f_o)$  by customary techniques.

One such technique - the partial fraction expansion of  $W(s; \rho_o, f_o)$  - provides the motivation for the following discussion.

Let it be assumed for simplicity that the echo to be detected has zero Doppler shift.<sup>†</sup> The case of non-zero Doppler can be studied in the manner to be outlined, but at the expense of extremely laborious algebra.

If in Eq. (12.14) one introduces  $M(s; \rho_o, 0)$  for the zero-Doppler echo, the result is

---

<sup>†</sup> This is the case considered by Urkowitz<sup>45</sup> for clutter alone.

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$W(s; \rho_o, 0) = \frac{T^{-\frac{1}{2}} s \cdot 2 \sinh\left(\frac{sT}{2}\right)}{N_o s^2 + \frac{2\mathcal{E}c}{T} \cdot 4 \sinh^2\left(\frac{sT}{2}\right)} \cdot e^{-s\rho_o} \quad (12.16)$$

No loss of generality results if  $\rho_o$  is now taken to be zero and attention is restricted to

$$W(s; 0, 0) = \frac{T^{-\frac{1}{2}} s \cdot 2 \sinh\left(\frac{sT}{2}\right)}{N_o s^2 + \frac{2\mathcal{E}c}{T} \cdot 4 \sinh^2\left(\frac{sT}{2}\right)} \quad (12.17)$$

The characteristic equation in either case is given by

$$D(s) = N_o s^2 + \frac{2\mathcal{E}c}{T} \cdot 4 \sinh^2\left(\frac{sT}{2}\right) = 0 \quad (12.18)$$

It may be shown that the roots of this characteristic equation are symmetrically disposed in quadruplets in the s-plane. This follows from the fact that, for complex s,

$$i) \quad (s)^2 \equiv (-s)^2 \quad \text{and} \quad \sinh^2\left(-\frac{sT}{2}\right) \equiv \left[-\sinh\left(\frac{sT}{2}\right)\right]^2 \quad (12.19a)$$

and

$$ii) \quad (s^2)^* \equiv (s^*)^2 \quad \text{and} \quad \sinh\left(\frac{s^*T}{2}\right) \equiv \left[\sinh\left(\frac{sT}{2}\right)\right]^* \quad (12.19b)$$

Thus, from these symmetry relations for the individual terms of  $D(0)$ , one concludes that, if for some  $\omega_k$  and  $\omega_k$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$i) \quad D(\sigma_k + j\omega_k) = 0, \quad (12.20a)$$

then also

$$ii) \quad D(-\sigma_k - j\omega_k) = 0, \quad (12.20b)$$

$$iii) \quad D(\sigma_k - j\omega_k) = 0, \quad (12.20c)$$

and

$$iv) \quad D(-\sigma_k + j\omega_k) = 0. \quad (12.20d)$$

This symmetric disposition of roots is shown in Fig. 19.

It will presently be shown that, for  $N_0 \neq 0$ , all the roots of the characteristic equation are simple and lie away from the  $\omega$ -axis (i.e., have  $\sigma \neq 0$ ). The expansion of  $W(s; 0, 0)$  in partial fractions will therefore have the form, for  $\sigma_k > 0$  and  $\omega_k > 0$ ,

$$W(s; 0, 0) = \sum_k \left\{ \frac{a_{k1}}{s - (\sigma_k + j\omega_k)} + \frac{a_{k2}}{s - (-\sigma_k + j\omega_k)} + \frac{a_{k3}}{s - (-\sigma_k - j\omega_k)} + \frac{a_{k4}}{s - (\sigma_k - j\omega_k)} \right\} \quad (12.21)$$

where  $a_{kq}$  is the residue for the  $k$ -th pole in the  $q$ -th quadrant of the  $s$ -plane. Finding the inverse transform of this partial fraction expansion is a relatively straightforward matter.

It is known<sup>9</sup>, and readily verified, that

$$\int_{-j\infty}^{+j\infty} \frac{a_0}{s - s_0} \cdot e^{st} \cdot ds = \begin{cases} a_0 \cdot e^{s_0 t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \quad (12.22a)$$

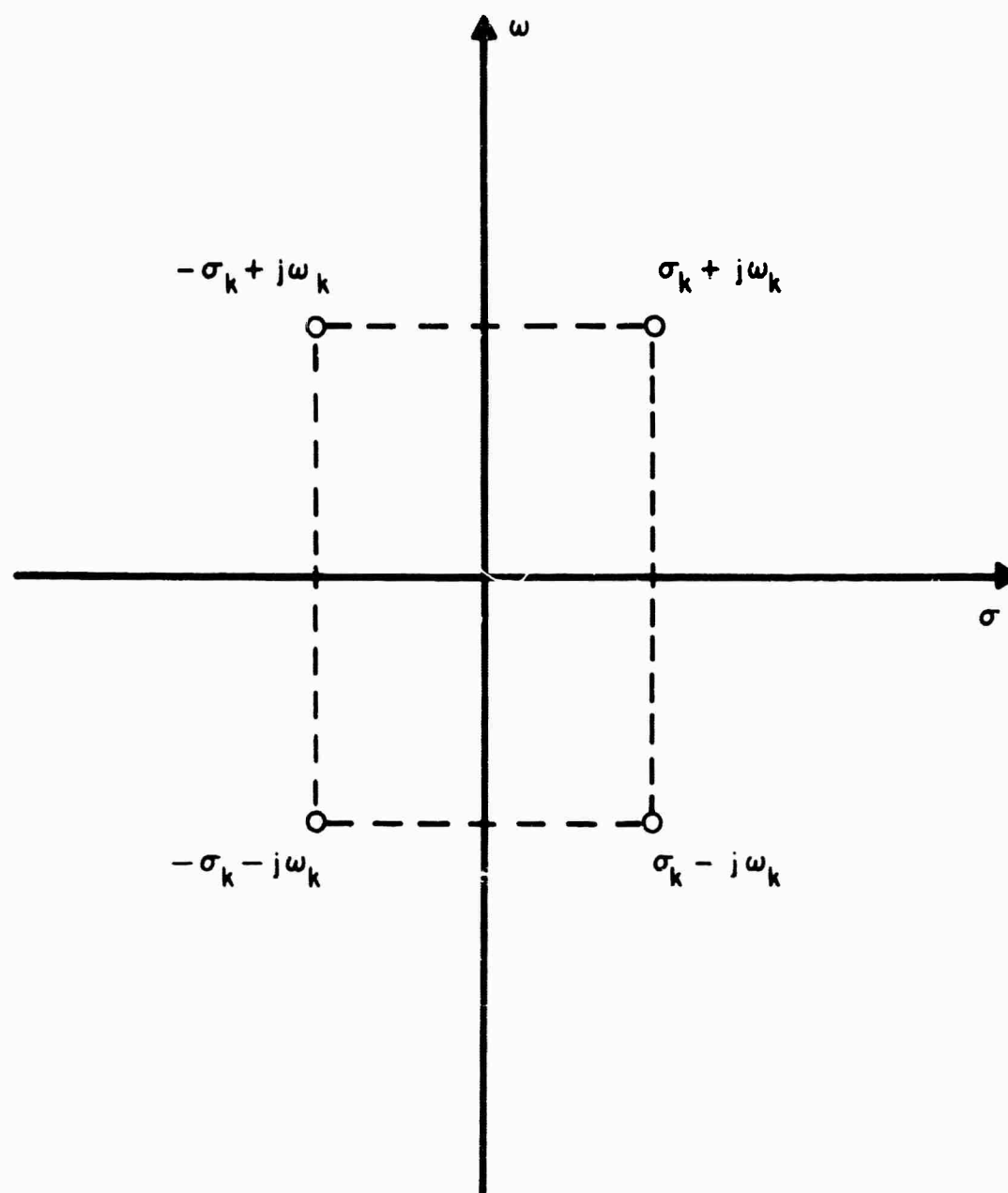


FIG. 19 A TYPICAL QUADRUPLT OF CHARACTERISTIC ROOTS

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

for  $s_0$  in the left half-plane. Conversely, for zeroes in the right half-plane, one may write

$$\int_{-j\infty}^{+j\infty} \frac{a_0}{s - s_0} e^{st} ds = \begin{cases} 0 & \text{for } t \geq 0 \\ -a_0 \cdot e^{s_0 t} & \text{for } t < 0 \end{cases} \quad (12.22b)$$

where  $\text{Re}\{s\} > 0$ . Verification of these relations is accomplished by direct substitution of the indicated time functions into Eq. (12.11b). When the integrals are evaluated, the resulting transforms will be seen to agree with the transforms under the integrals in Eq. (12.22).

When Eqs. (12.21) and (12.22) are compared, it may be seen that the time function which is the inverse transform of  $W(s;0,0)$  may be written in the form

$$w(t;0,0) = \sum_k \{w_k^+(t) + w_k^-(t)\} \quad (12.23a)$$

where

$$w_k^+(t) = \begin{cases} a_{k2} \cdot e^{-\sigma_k t} e^{j\omega_k t} + a_{k3} \cdot e^{-\sigma_k t} e^{-j\omega_k t}, & t > 0 \\ 0, & \text{for } t < 0; \end{cases} \quad (12.23b)$$

$$(12.23c)$$

and

$$w_k^-(t) = \begin{cases} 0, & \text{for } t > 0 \\ -a_{k4} e^{\sigma_k t} e^{-j\omega_k t} - a_{k1} e^{\sigma_k t} e^{j\omega_k t}, & t < 0. \end{cases} \quad (12.23d)$$

$$(12.23e)$$

The only remaining problems are finding the zeroes  $(\sigma_k, \omega_k)$  and evaluating the corresponding residues  $a_{kj}$ .

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

To find the roots of  $D(s)$ , first observe that  $D(s)$  may be factored and the characteristic Eq. (12.18) written

$$D(s) = \left\{ N_0^{\frac{1}{2}} \cdot s + j \left( \frac{2\mathcal{E}c}{T} \right)^{\frac{1}{2}} \cdot 2 \sinh\left(\frac{sT}{2}\right) \right\} \\ \cdot \left\{ N_0^{\frac{1}{2}} \cdot s - j \left( \frac{2\mathcal{E}c}{T} \right)^{\frac{1}{2}} \cdot 2 \sinh\left(\frac{sT}{2}\right) \right\} = 0 . \quad (12.24)$$

The roots of  $D(s)$  are then values of  $s$  such that either

$$N_0^{\frac{1}{2}} \cdot s + j \left( \frac{2\mathcal{E}c}{T} \right)^{\frac{1}{2}} \cdot 2 \sinh\left(\frac{sT}{2}\right) = 0 \quad (12.25a)$$

or

$$N_0^{\frac{1}{2}} \cdot s - j \left( \frac{2\mathcal{E}c}{T} \right)^{\frac{1}{2}} \cdot 2 \sinh\left(\frac{sT}{2}\right) = 0 . \quad (12.25b)$$

At this point a certain economy may be effected by recalling Eqs. (12.19) and noting that if  $s_0$  satisfies Eq. (12.25a), then  $-s_0$  will also satisfy Eq. (12.25a) while  $s_0^*$  satisfies Eq. (12.25b). It is therefore sufficient to consider only the roots of Eq. (12.25a) alone<sup>†</sup>, in order to discover all the roots of  $D(s)$ . Substitution of

$$s = \sigma + j\omega \quad (12.26)$$

into Eq. (12.25a), followed by expansion of the hyperbolic sine, therefore leads to

$$N_0^{\frac{1}{2}} \sigma - \left( \frac{2\mathcal{E}c}{T} \right)^{\frac{1}{2}} \cdot 2 \cosh\left(\frac{\sigma T}{2}\right) \sin\left(\frac{\omega T}{2}\right) = 0 \quad (12.27a)$$

---

<sup>†</sup> Or Eq. (12.25b) alone.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

and

$$N_0^{\frac{1}{2}} \omega + \left( \frac{2\mathcal{E}_c}{T} \right)^{\frac{1}{2}} \cdot 2 \sinh\left(\frac{\sigma T}{2}\right) \cos\left(\frac{\omega T}{2}\right) = 0 \quad (12.27b)$$

as a pair of equations to be solved simultaneously for  $\sigma$  and  $\omega$ .

It would seem that, in general, solutions to Eqs. (12.27) are to be had only by approximate or numerical techniques. The asymptotic behavior of the solutions as  $N_0$  approaches zero will in fact be considered in the next section. A general indication of the root locations may, however, be obtained in the following manner.

From Eqs. (12.27) one can derive the following auxiliary relation which must be satisfied by any solution  $(\sigma, \omega)$  of Eqs. (12.27):

$$\left(\frac{\omega T}{2}\right) \cdot \tan\left(\frac{\omega T}{2}\right) = -\left(\frac{\sigma T}{2}\right) \tanh\left(\frac{\sigma T}{2}\right) \quad (12.28)$$

The left and right sides of this equation are plotted separately, and to the left and right, in Fig. 20. Each term is seen to be an even function of its argument.

Consider now the interval

$$l\pi - \frac{\pi}{2} < \frac{\omega T}{2} < l\pi \quad (12.29)$$

where  $l$  is a positive integer. Clearly, for each  $\omega$  in this interval, there is a  $\sigma$  in the interval

$$0 < \left(\frac{\sigma T}{2}\right) < \infty \quad (12.30)$$

which causes condition (12.28) to be satisfied. Over this same interval

$$0 < \sinh\left(\frac{\sigma T}{2}\right) < \infty \quad (12.31)$$



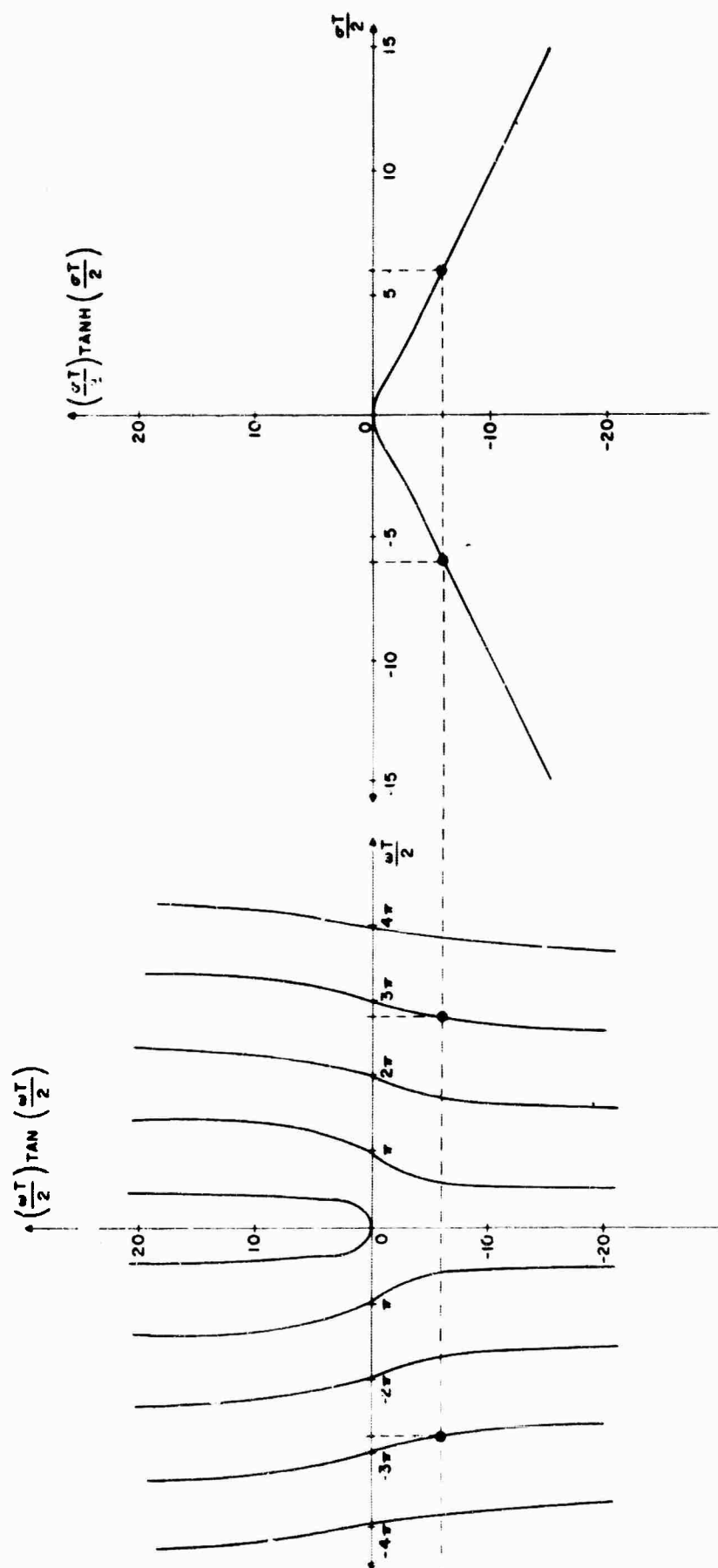


FIG. 20 FUNCTIONS DETERMINING LOCATIONS OF CHARACTERISTIC ROOTS

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

while  $\cos(\frac{\omega T}{2})$  has fixed sign, given by

$$\text{sgn} \cos(\frac{\omega T}{2}) = (-1)^l \quad (12.32)$$

Consider now the variation of the left member of Eq. (12.27b) as  $\frac{\omega T}{2}$  ranges upward over the interval defined by the inequalities (12.29). The first term,  $N_0^2 \omega$ , remains positive and increases. The second term varies† from zero to  $(-1)^l(+\infty)$ . Thus if  $l$  is odd, the left member of Eq. (12.27b) must undergo a sign change in the interval. Therefore, for  $l$  odd, there must be a root  $(\sigma, \omega)$  such that

$$l\pi - \frac{\pi}{2} < \frac{\omega T}{2} < l\pi \quad (12.33a)$$

and

$$0 < \frac{\sigma T}{2} < \infty. \quad (12.33b)$$

By exactly analogous reasoning one establishes that for even  $l$ , the root  $(\sigma, \omega)$  is such that

$$l\pi - \frac{\pi}{2} < \frac{\omega T}{2} < l\pi \quad (12.33c)$$

while

$$-\infty < \frac{\sigma T}{2} < 0 \quad (12.33d)$$

The inequalities (12.33), together with the symmetry relations (12.20) for the roots of  $D(s)$ , therefore lead to the conclusion that: for each positive, integral  $k$ , there exists a root of  $D(s)$  called  $s_k$ , such that

$$s_k = \sigma_k + j\omega_k \quad (12.34a)$$

---

† As deduced from Eqs. (12.31) and (12.32).

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

where 
$$0 < \frac{\sigma_k \cdot T}{2} \quad (12.34b)$$

$$k\pi - \frac{\pi}{2} < \frac{\omega_k \cdot T}{2} < k\pi \quad (12.34c)$$

and 
$$D(\sigma_k + j\omega_k) = 0 \quad (12.34d)$$

$$D(-\sigma_k + j\omega_k) = 0 \quad (12.34e)$$

$$D(-\sigma_k - j\omega_k) = 0 \quad (12.34f)$$

$$D(\sigma_k - j\omega_k) = 0 \quad (12.34g)$$

That there exists only one such simple root  $s_k$  for each value of  $k$  will be shown by a continuity argument in the next section of this chapter.

Assuming then that a satisfactory collection of roots  $s_k$  has been determined, the final data needed for computation of Eqs. (12.23) are the four residues  $a_{kq}$  for each value of  $k$ . Because the roots are simple the algebra is straightforward.

Let  $W(s;0,0)$  given by Eq. (12.17) be written in the form

$$W(s;0,0) = \frac{N(s)}{D(s)} \quad (12.35)$$

where the numerator and denominator of Eq. (12.17) have been identified respectively with  $N(s)$  and  $D(s)$ . Then, if the individual roots of the  $k$ -th quadruplet are identified as to their quadrant by the subscript  $q$ , one may write<sup>†</sup>

<sup>†</sup> See Churchill,<sup>9</sup> pp. 57-58, for example.

$$a_{kq} = \left. \frac{N(s)}{D'(s)} \right]_{s = s_{kq}} \quad (12.36)$$

where the prime indicates differentiation with respect to  $s$ .

If the calculation called for by Eq. (12.36) is carried out, one has

$$D'(s) = 2N_0 \cdot s + 2 \cdot \frac{2\epsilon_c}{T} \cdot 4 \sinh\left(\frac{sT}{2}\right) \cdot \cosh\left(\frac{sT}{2}\right) \cdot \frac{T}{2} \quad (12.37)$$

which is the same as

$$D'(s) = 2N_0 \cdot s + 2\epsilon_c \cdot 2 \sinh(sT). \quad (12.38)$$

If the real and imaginary parts of  $s$  are introduced, then  $N(s)$  and  $D'(s)$  may be separated into real and imaginary components, and the residue  $a_{kq}$  written

$$a_{kq} = \left. \frac{n_1(\sigma, \omega) + jn_2(\sigma, \omega)}{d_1(\sigma, \omega) + jd_2(\sigma, \omega)} \right]_{\substack{\sigma = \sigma_{kq} \\ \omega = \omega_{kq}}} \quad (12.39)$$

where

$$n_1(\sigma, \omega) = \frac{2\sigma}{T^{\frac{1}{2}}} \sinh\left(\frac{\sigma T}{2}\right) \cos\left(\frac{\omega T}{2}\right) - \frac{2\omega}{T^{\frac{1}{2}}} \cosh\left(\frac{\sigma T}{2}\right) \sin\left(\frac{\omega T}{2}\right) \quad (12.40a)$$

$$n_2(\sigma, \omega) = \frac{2\omega}{T^{\frac{1}{2}}} \sinh\left(\frac{\sigma T}{2}\right) \cos\left(\frac{\omega T}{2}\right) + \frac{2\sigma}{T^{\frac{1}{2}}} \cosh\left(\frac{\sigma T}{2}\right) \sin\left(\frac{\omega T}{2}\right) \quad (12.40b)$$

$$d_1(\sigma, \omega) = 2N_0 \sigma + 2\epsilon_c \cdot 2 \sinh(\sigma T) \cos(\omega T) \quad (12.40c)$$

$$d_2(\sigma, \omega) = 2N_0\omega + 2\epsilon_c \cdot 2 \cosh(\sigma T) \sin(\omega T) \quad (12.40d)$$

Finally  $a_{kq}$  may be separated into its real and imaginary components. Thus

$$a_{kq} = \frac{m_1 \cdot d_1 + m_2 \cdot d_2}{d_1^2 + d_2^2} + j \frac{m_2 \cdot d_1 - m_1 \cdot d_2}{d_1^2 + d_2^2} \quad (12.41)$$

where the functions defined in Eqs. (12.40) have been used with abbreviated notation.

Before concluding this section on general aspects of the solution, two simplifying remarks may be made. First, one sees by inspection of Eq. (12.17) that, for all  $s$ ,

$$W(s; 0, 0) \equiv W(-s; 0, 0) \quad (12.42)$$

If in the basic inversion formula, Eq. (12.11a), one changes variable according to  $r = -s$  and then uses Eq. (12.42), then one may show directly that for all  $t$

$$w(t; 0, 0) \equiv w(-t; 0, 0) \quad (12.43)$$

Second, one may consider the two roots  $(-\sigma_k + j\omega_k)$  and  $(-\sigma_k - j\omega_k)$  and trace the effects of the change of sign of  $\omega_k$  through Eqs. (12.40) and (12.41) down to the conclusion that

$$a_{k2} = a_{k3}^* \quad (12.44)$$

Both Eqs. (12.43) and (12.44) now enable one to write more simply, and in place of Eqs. (12.23),

$$w(t;0,0) = 2 \sum_{k=1}^{\infty} \operatorname{Re} \left\{ a_{k2} e^{-\sigma_k t} e^{j\omega_k t} \right\}, \quad t > 0 \quad (12.45a)$$

$$w(t;0,0) = w(-t;0,0), \quad t < 0 \quad (12.45b)$$

where  $\sigma_k > 0$  and  $\omega_k > 0$ .

### C. SMALL-NOISE SOLUTION

For  $N_0$  approaching zero it is possible to derive approximate expressions for the roots of the characteristic equation and, therefore, ultimately to compute approximations to  $w(t;0,0)$  given by Eq. (12.45a). This approximate analysis for small noise is the subject of this section.

The starting point for the analysis is the pair of Eqs. (12.27a) and (12.27b). When  $N_0$  is actually zero they become

$$-\left(\frac{2\mathcal{E}_c}{T}\right)^{\frac{1}{2}} \cdot 2 \cosh\left(\frac{\sigma T}{2}\right) \sin\left(\frac{\omega T}{2}\right) = 0 \quad (12.45a)$$

and

$$\left(\frac{2\mathcal{E}_c}{T}\right)^{\frac{1}{2}} \cdot 2 \sinh\left(\frac{\sigma T}{2}\right) \cos\left(\frac{\omega T}{2}\right) = 0 \quad (12.45b)$$

The simultaneous solution of this simpler pair of equations presents few problems. Since for all real  $x$ ,  $\cosh x$  is greater than zero, the first equation can only be satisfied if

$$\sin\left(\frac{\omega T}{2}\right) = 0 \quad (12.46a)$$

or

$$\frac{\omega T}{2} = l\pi, \quad l = 1, 2, 3, \dots^\dagger \quad (12.46b)$$

---

<sup>†</sup> Only positive arguments need be considered in view of Eq. (12.45a).

Since the cosine has unit magnitude for these arguments, Eq. (12.45b) therefore implies that

$$\frac{\sigma T}{2} = 0 \quad (12.47)$$

for all solutions.

These zero-noise solutions are the starting point for finding the small-noise, approximate solutions. Note in passing, however, that the zeroes defined by Eqs. (12.46) and (12.47) are actually double zeroes of the characteristic equation. It is readily verified that for any individual  $(\sigma, \omega)$  equal to  $(0, \frac{2l\pi}{T})$ , both of the factors of  $D(s)$  in Eq. (12.24) are equal to zero. Each quadruplet of simple zeroes for  $N_0 \neq 0$  therefore originates as a pair of double zeroes when  $N_0 = 0$ .

For  $N_0$  sufficiently small one expects that the zeroes of  $D(s)$ , Eq. (12.18), will be near the zeroes for  $N_0$  equal to zero. With this expectation, let

$$\omega_l = \frac{2\pi l}{T} + \delta\omega_l \equiv \omega_{l0} + \delta\omega_l \quad (12.48)$$

and

$$\sigma_l = 0 + \delta\sigma_l \quad (12.49)$$

Substitution into Eqs. (12.27) then leads to

$$N_0^{\frac{1}{2}} \cdot \delta\sigma_l - \left(\frac{2\mathcal{E}_c}{T}\right)^{\frac{1}{2}} \cdot 2 \cosh\left(\frac{\delta\sigma_l T}{2}\right) \cdot (-1)^l \sin \frac{\delta\omega_l T}{2} = 0 \quad (12.50a)$$

and

$$N_o^{\frac{1}{2}}(\omega_{lo} + \delta\omega_l) + \left(\frac{2\epsilon_c}{T}\right)^{\frac{1}{2}} \cdot 2 \sinh\left(\frac{\delta\sigma_l T}{2}\right) \cdot (-1)^l \cos\left(\frac{\delta\omega_l T}{2}\right) = 0 \quad (12.50b)$$

For the "small"  $\delta\sigma_l$  and  $\delta\omega_l$  which result for  $N_o$  sufficiently small, these equations may be replaced by the approximate pair

$$N_o^{\frac{1}{2}} \delta\sigma_l - \left(\frac{2\epsilon_c}{T}\right)^{\frac{1}{2}} \cdot 2 \cdot (-1)^l \cdot \left(\frac{\delta\omega_l T}{2}\right) = 0 \quad (12.51a)$$

and

$$N_o^{\frac{1}{2}} \delta\omega_l + \left(\frac{2\epsilon_c}{T}\right)^{\frac{1}{2}} \cdot 2 \cdot \left(\frac{\delta\sigma_l T}{2}\right) \cdot (-1)^l \cdot 1 = -N_o^{\frac{1}{2}} \omega_{lo} \quad (12.51b)$$

Simultaneous solution yields

$$\delta\sigma_l = \left(\frac{2\epsilon_c}{N_o T}\right)^{\frac{1}{2}} \cdot 2 \cdot (-1)^l \cdot \left(\frac{\delta\omega_l T}{2}\right) \quad (12.52a)$$

and

$$\delta\omega_l = - \frac{N_o}{N_o + \frac{2\epsilon_c}{T} \cdot 2^2 \cdot (-1)^{2l} \cdot \left(\frac{T}{2}\right)^2} \cdot \omega_{lo} \quad (12.52b)$$

For  $N_o$  approaching zero, this last equation yields

$$\delta\omega_l \doteq -1 \cdot \left(\frac{N_o}{2\epsilon_c T}\right) \frac{2\pi l}{T} \quad (12.53a)$$

which, when substituted into Eq. (12.52a), yields in turn

$$\delta\sigma_l \doteq -1 \cdot (-1)^l \cdot \left(\frac{N_o}{2\epsilon_c T}\right)^{\frac{1}{2}} \cdot \frac{2\pi l}{T} \quad (12.53b)$$

These equations form the basis for Fig. 21 showing root loci near the  $\sigma$ -axis for small  $N_o$ . From Eqs. (12.53) one finds



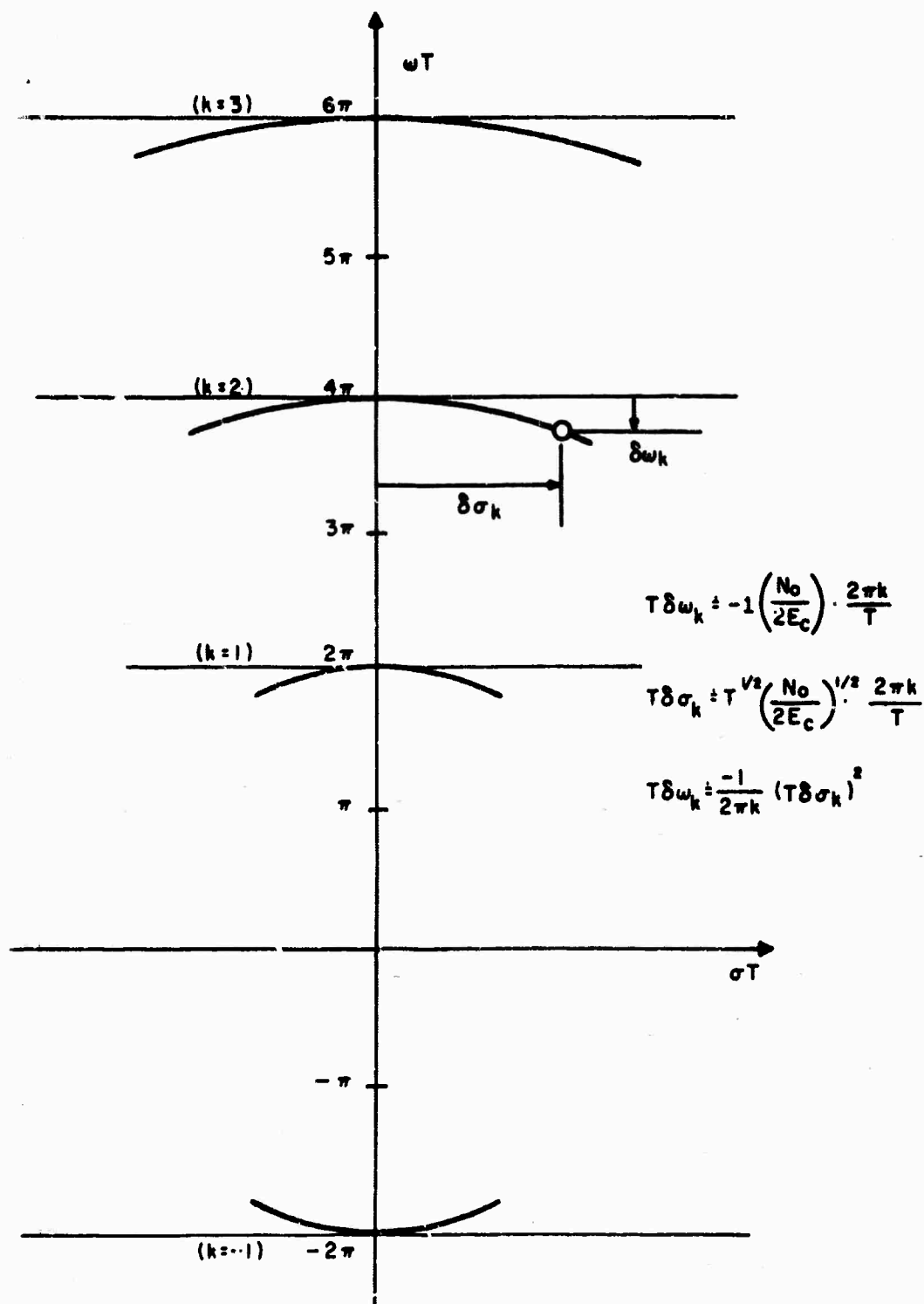


FIG. 21 CHARACTERISTIC ROOT LOCI FOR SMALL  $N_0$ .

$$\delta\omega_l = -1 \cdot \frac{T}{2\pi l} \cdot (\delta\sigma_l)^2 \quad l = 1, 2, 3, \dots \quad (12.54)$$

as the equation for the indicated segments of the root locus curves. The actual root location on any particular curve will depend upon the parameter  $\left(\frac{N_0}{2\sigma_c T}\right)$ , via Eqs. (12.53). This question will be considered in a moment.

Here, however, it is convenient to pause temporarily to note that as  $N_0$  increases from zero, each of the double roots of  $D(s)$  located, for  $N_0 = 0$ , at

$$s_l = \pm j \frac{2\pi l}{T} \quad l = 1, 2, 3, \dots \quad (12.55)$$

splits and gives rise to a single root in each of the quadrants, with imaginary parts satisfying

$$|\omega_l| < \frac{2\pi l}{T} \quad l = 1, 2, 3, \dots \quad (12.56)$$

Reverting now to consideration of the inequalities (12.33) which are strictly satisfied for all  $N_0 > 0$ , one concludes that, for each  $l$ , there is only a single root which satisfies Eqs. (12.34). Since this conclusion is true for sufficiently small  $N_0$ , it remains true for large  $N_0$ . Viewed differently, one can say that the root locus which intersects the  $\omega$ -axis at

$$\omega_{l0} = \frac{2\pi l}{T} \quad l = 1, 2, 3, \dots \quad (12.57)$$

is constrained to lie in the band in the  $s$ -plane defined by the (strict) inequalities

$$\frac{2\pi l}{T} - \frac{\pi}{T} < \omega < \frac{2\pi l}{T} \quad (12.58)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The approximate root locations for a given  $N_0$  may be estimated from Eqs. (12.53b). It is clear, however, that for sufficiently large  $l$ , neither  $\delta\sigma_l$  nor  $\delta\omega_l$  will be small for  $N_0$  fixed. For large  $l$ , therefore, the accuracy of the approximations (12.53) can be doubted.

A possibly better approximation to the root locations for large  $l$  rests upon the inequalities (12.58). These bounds imply that

$$\frac{\omega_l T}{l} \rightarrow 2\pi \quad (12.59)$$

as  $l \rightarrow \infty$ , or that  $\omega_l$  is of the order of  $l$ . Consider now Eq. (12.27b). If  $\cos \frac{\omega_l T}{2}$  does not approach zero too rapidly as  $l \rightarrow \infty$ , then Eq. (12.27b) suggests that

$$\omega_l = \sigma \left\{ \left( \frac{2\epsilon_c}{N_0 T} \right)^{\frac{1}{2}} \cdot 2 \sinh \left( \frac{\sigma_l T}{2} \right) \right\} \quad (12.60)$$

These considerations are summarized in Fig. 22. On the fundamental strips implied by Eq. (12.58) are superimposed the straight line implied by Eq. (12.53b) and the hyperbolic sine appearing in Eq. (12.60). The circles indicate actual root locations which satisfy the exact characteristic Eq. (12.25).†

The fact of probably greatest theoretical interest is that, for all  $l$ , according to Eq. (12.53b),

$$\sigma_l \propto \left( \frac{N_0}{2\epsilon_c T} \right)^{\frac{1}{2}} \quad (12.61)$$

Because  $\sigma_l$  varies as the square-root of  $N_0$ , even relatively "small" values of  $N_0$  lead to relatively "large" displacements

† The indicated roots were found by successive approximations using a digital computer for numerical calculations.

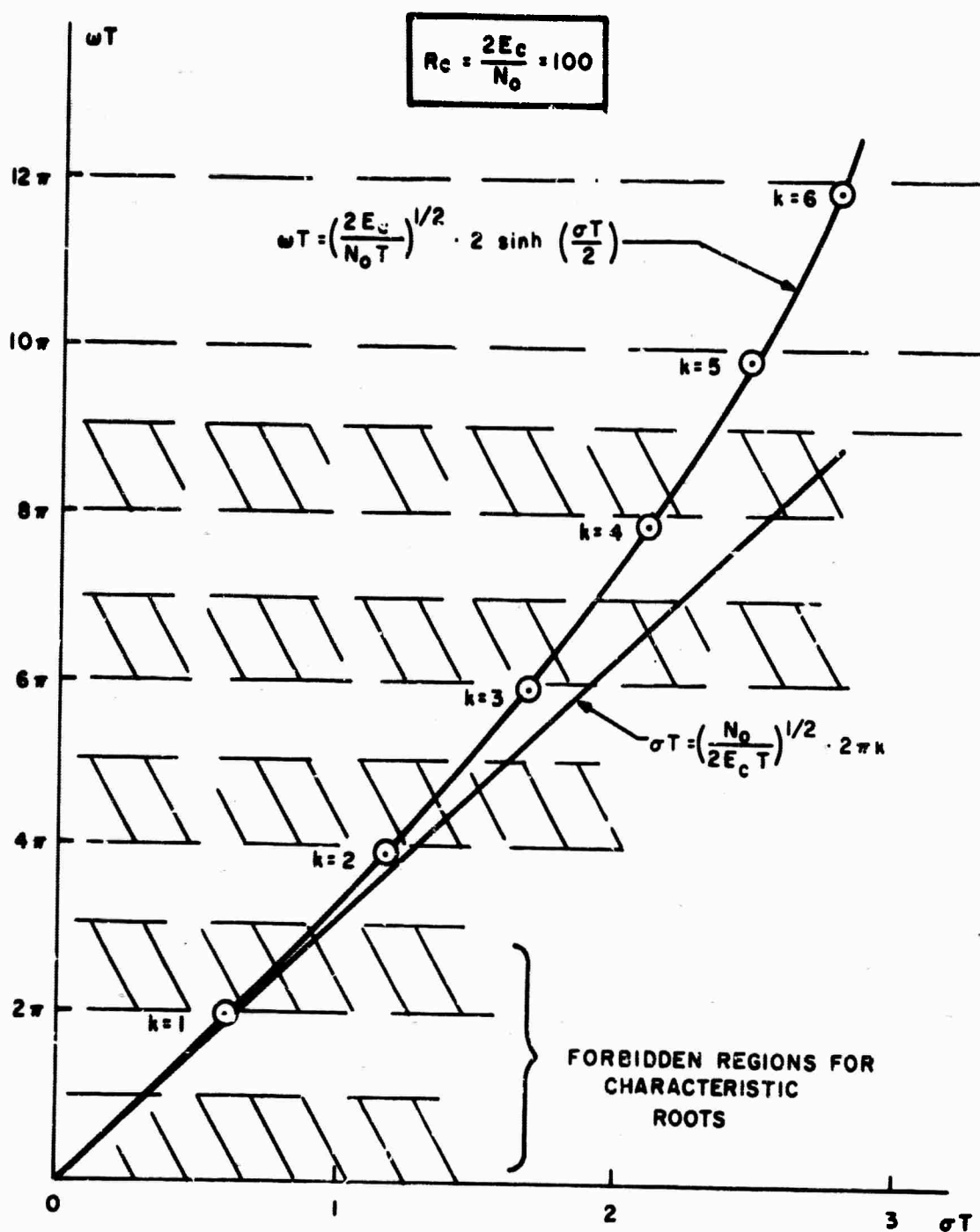


FIG. 22a CHARACTERISTIC ROOTS FOR SMALL  $N_0$  (FIRST QUADRANT; LARGE SCALE)

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

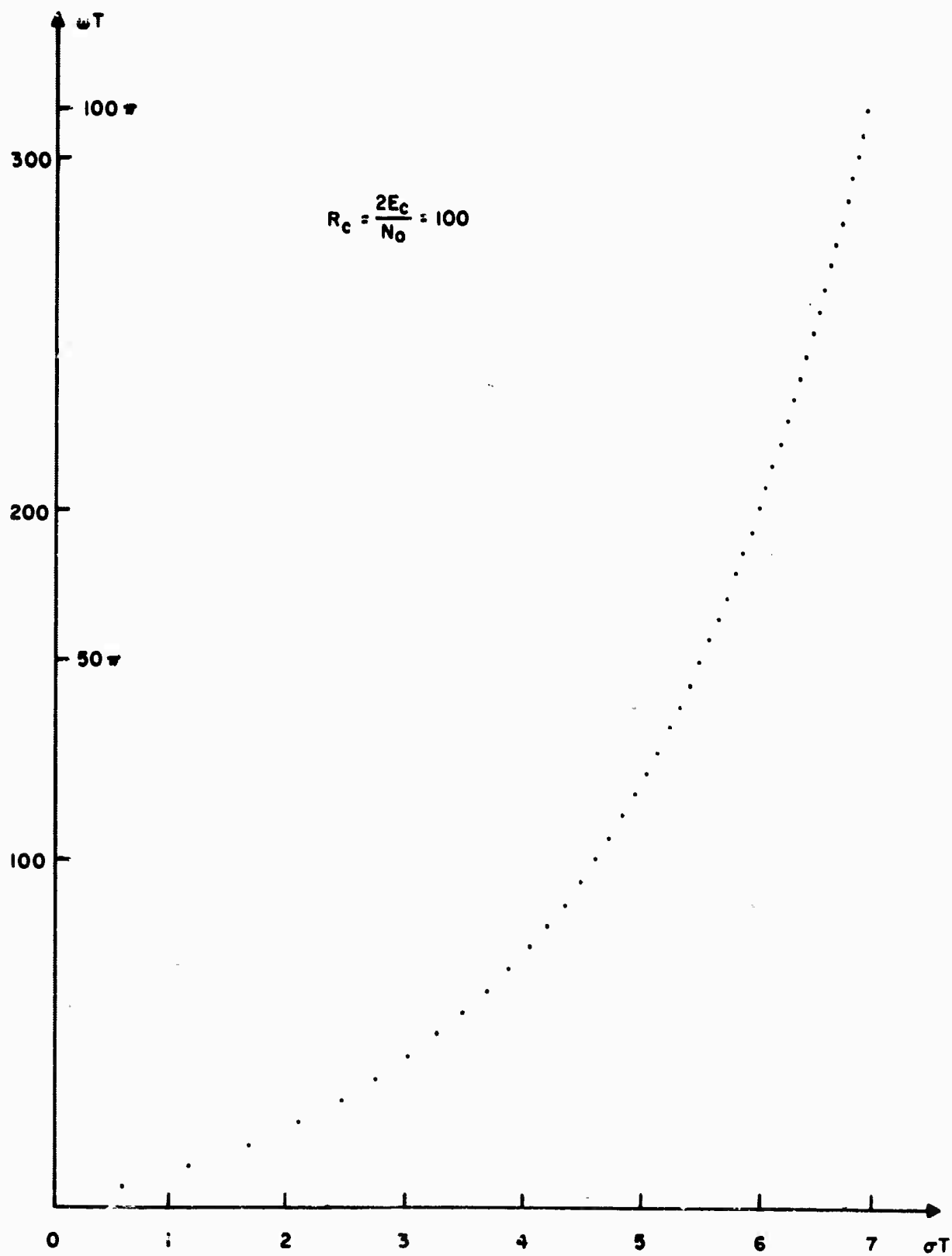


FIG. 22b CHARACTERISTIC ROOTS FOR SMALL  $N_0$ . (FIRST QUADRANT; SMALL SCALE)

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

of the characteristic root locations from their zero-noise locations. The effect will be seen to cause analogous relatively "large" deviations of the small-noise problem solution from the no-noise solution. This is seen by considering the asymptotic behavior of the solution, Eq. (12.45) as  $N_o \rightarrow 0$ .

The asymptotic behavior of the residue  $a_{k2}$  is found by substituting the values

$$\sigma_{k2} = -\delta\sigma_k, \quad \delta\sigma_k < 0 \quad (12.62a)$$

and

$$\omega_{k2} = \frac{2\pi k}{T} + \delta\omega_k, \quad \delta\omega_k < 0 \quad (12.62b)$$

into Eqs. (12.40) to find that

$$n_1(\sigma_{k2}, \omega_{k2}) \doteq (-1)^k \left\{ T^{\frac{1}{2}} \cdot (\delta\sigma_k)^2 - \left( \frac{2\pi k}{T} \right) \cdot T^{\frac{1}{2}} \cdot (\delta\omega_k) \right\} \quad (12.63a)$$

$$n_2(\sigma_{k2}, \omega_{k2}) \doteq (-1)^k \left\{ -\left( \frac{2\pi k}{T} \right) \cdot T^{\frac{1}{2}} \cdot (\delta\sigma_k) - (\delta\sigma_k) \cdot T^{\frac{1}{2}} \cdot (\delta\omega_k) \right\} \quad (12.63b)$$

$$d_1(\sigma_{k2}, \omega_{k2}) \doteq 2 \left\{ -N_o (\delta\sigma_k) - 2\mathcal{E}_c \cdot T \cdot (\delta\sigma_k) \right\} \quad (12.63c)$$

$$d_2(\sigma_{k2}, \omega_{k2}) \doteq 2 \left\{ N_o \left( \frac{2\pi k}{T} \right) + 2\mathcal{E}_c \cdot T \cdot (\delta\omega_k) \right\} \quad (12.63d)$$

For a second quadrant zero, one uses Eqs. (12.53) to write (consistent with Eqs. 12.62)

$$\delta\sigma_k \doteq + \left( \frac{N_o}{2\mathcal{E}_c T} \right)^{\frac{1}{2}} \cdot \frac{2\pi k}{T} \quad (12.64a)$$

and

$$\delta\omega_k \doteq -\left(\frac{N_o}{2\epsilon_c T}\right) \cdot \frac{2\pi k}{T} \quad (12.64b)$$

Equations (12.63) may therefore be reduced to

$$n_1(\sigma_{k2}, \omega_{k2}) \doteq 2(-1)^k \cdot \left(\frac{N_o}{2\epsilon_c T}\right) \left(\frac{2\pi k}{T}\right)^2 \cdot T^{\frac{1}{2}} \quad (12.65a)$$

$$n_2(\sigma_{k2}, \omega_{k2}) \doteq -(-1)^k \cdot \left(\frac{N_o}{2\epsilon_c T}\right)^{\frac{1}{2}} \left(\frac{2\pi k}{T}\right)^2 \cdot T^{\frac{1}{2}} \quad (12.65b)$$

$$d_1(\sigma_{k2}, \omega_{k2}) \doteq -2 \{N_o + 2\epsilon_c T\} \cdot \left(\frac{N_o}{2\epsilon_c T}\right)^{\frac{1}{2}} \left(\frac{2\pi k}{T}\right) \quad (12.65c)$$

$$d_2(\sigma_{k2}, \omega_{k2}) \doteq 2 \{N_o\} \cdot 1 \cdot \left(\frac{2\pi k}{T}\right) \quad (12.65d)$$

When these values are finally introduced into Eq. (12.41), the result for the residue  $a_{k2}$  is

$$a_{k2} \doteq (-1)^k \left(\frac{2\pi k}{T}\right) \left(\frac{1}{2\epsilon_c}\right) \left\{ -\frac{3}{2} \left(\frac{N_o}{2\epsilon_c T}\right)^{\frac{1}{2}} + j\left(\frac{1}{2}\right) \right\} \cdot T^{-\frac{1}{2}} \quad (12.66)$$

Finally, the solution for  $N_o$  approaching zero is found by using Eqs. (12.45) and (12.66). The result is, for  $t > 0$ ,

$$w_{opt}(t; 0, 0) \doteq \frac{T^{-\frac{1}{2}}}{2\epsilon_c} \sum_{k=1}^{\infty} (-1)^k \left(\frac{2\pi k}{T}\right) e^{-\sigma_k t} \left\{ -\sin(\omega_k t) - \frac{3}{2} \left(\frac{N_o}{2\epsilon_c T}\right)^{\frac{1}{2}} \cos(\omega_k t) \right\} \quad (12.67a)$$

$$w_{opt}(-t; 0, 0) = w_{opt}(t; 0, 0) \quad (12.67b)$$

where

$$\sigma_k \doteq \frac{2\pi k}{T} \left(\frac{N_o}{2\epsilon_c T}\right)^{\frac{1}{2}} \quad (12.67c)$$

and 
$$\omega_k \doteq \frac{2\pi k}{T} \left\{ 1 - \left( \frac{N_0}{2\epsilon_c T} \right) \right\} \quad (12.67d)$$

The zero-noise solution, which is valuable for comparison purposes, can be deduced from the preceding expressions. Thus for  $N_0 = 0$  and  $t > 0$ ,

$$w_{\text{opt}}(t; 0, 0) = \frac{T^{-\frac{1}{2}}}{2\epsilon_c} \sum_{k=1}^{\infty} (-1)^k \left( \frac{2\pi k}{T} \right) \left\{ -\sin \left( \frac{2\pi k}{T} \cdot t \right) \right\} \quad (12.68)$$

which may be rewritten

$$w_{\text{opt}}(t; 0, 0) = \frac{T^{-\frac{1}{2}}}{2\epsilon_c} \cdot \frac{d}{dt} \sum_{k=1}^{\infty} \cos \left\{ \frac{2\pi k}{T} \left( t - \frac{T}{2} \right) \right\}. \quad (12.69)$$

The summation now is a recognizable form since, using only the sifting property of the impulse, one may write the following Fourier series expansion of an impulse train:

$$T \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT) = 1 + \sum_{k=1}^{\infty} \cos \left\{ \frac{2\pi k}{T} t \right\} \quad (12.70)$$

Equation (12.69) may therefore be written in the equivalent form, for  $t > 0$ ,

$$w_{\text{opt}}(t; 0, 0) = \frac{1}{2\epsilon_c} \frac{T^{\frac{1}{2}}}{2} \sum_{k=0}^{\infty} \frac{d}{dt} \delta \left( t - \frac{T}{2} - kT \right). \quad (12.71a)$$

For  $t < 0$ , Eq. (12.67b) remains unchanged

$$w_{\text{opt}}(-t; 0, 0) = w_{\text{opt}}(t; 0, 0) \quad (12.71b)$$



1. Discussion

The zero-noise solution of Eqs. (12.71) is illustrated in Fig. 23, in relation to the rectangular pulse to be detected. It is observed to be a sequence of doublets, spaced by the pulse duration and located to act upon the leading and trailing edges of the pulse. A physical insight into why the zero-noise weight function has these characteristics, and how they are related to the clutter dispersion function, will be given in the next section. For the present note simply that (for  $N_0 = 0$ ) the weight function extends, undiminished, to infinity in both directions.

The presence of a relatively small noise level, however, introduces the relatively large changes to be expected on the basis of Eqs. (12.67) and actually seen in Fig. 24. The weight function tends to become more restricted to the time of occurrence of the echo because of the exponential factor,  $\exp \{-\sigma_k t\}$ , appearing in Eq. (12.67a). The decay rate, governed by  $\sigma_k$ , moreover varies as the square-root of the noise level.

The doublets which characterized the zero-noise solution of Fig. 23 are seen to be very much "smoother" in the small noise solution of Fig. 24. The doublet nature of the solution is still quite pronounced in the vicinity of  $t = \pm 0.5$ , where the edges of the echo waveform are located. However, at the times  $t = \pm 1.5$  and  $t = \pm 2.5$  only faint suggestions of a doublet waveform exist.

The reason for the relative smoothness of the small-noise solution of Fig. 24 is the same exponential factor,  $\exp \{-\sigma_k t\}$ , which was mentioned two paragraphs back and which applies to every term of the solution in Eq. (12.67a). In the present context, however, one notes from Eq. (12.67c)

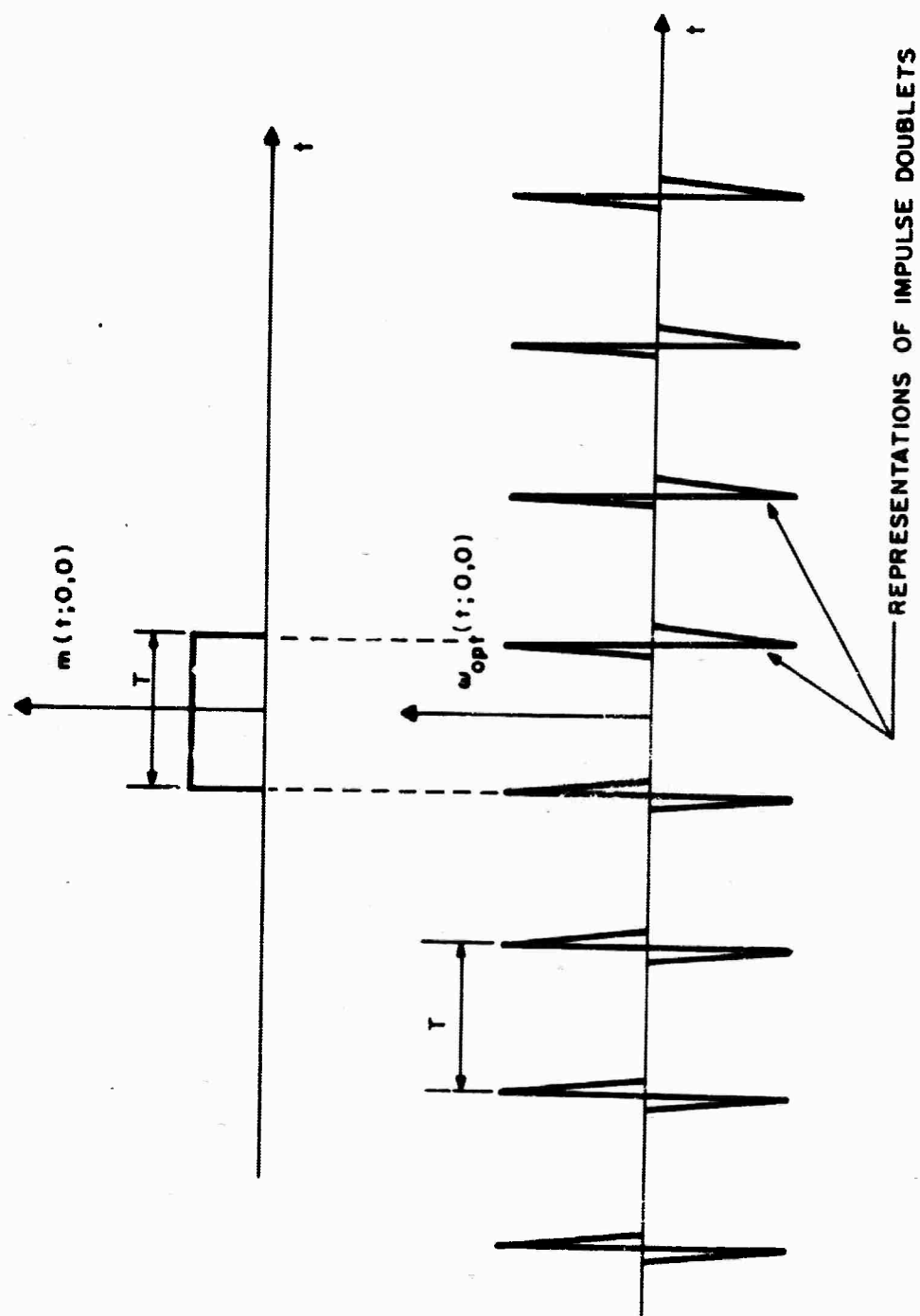


FIG. 23 OPTIMUM ZERO-NOISE PROCESSOR FOR A RECTANGULAR PULSE IN UNIFORMLY EXTENDED CLUTTER FROM A STAT:ONARY SOURCE

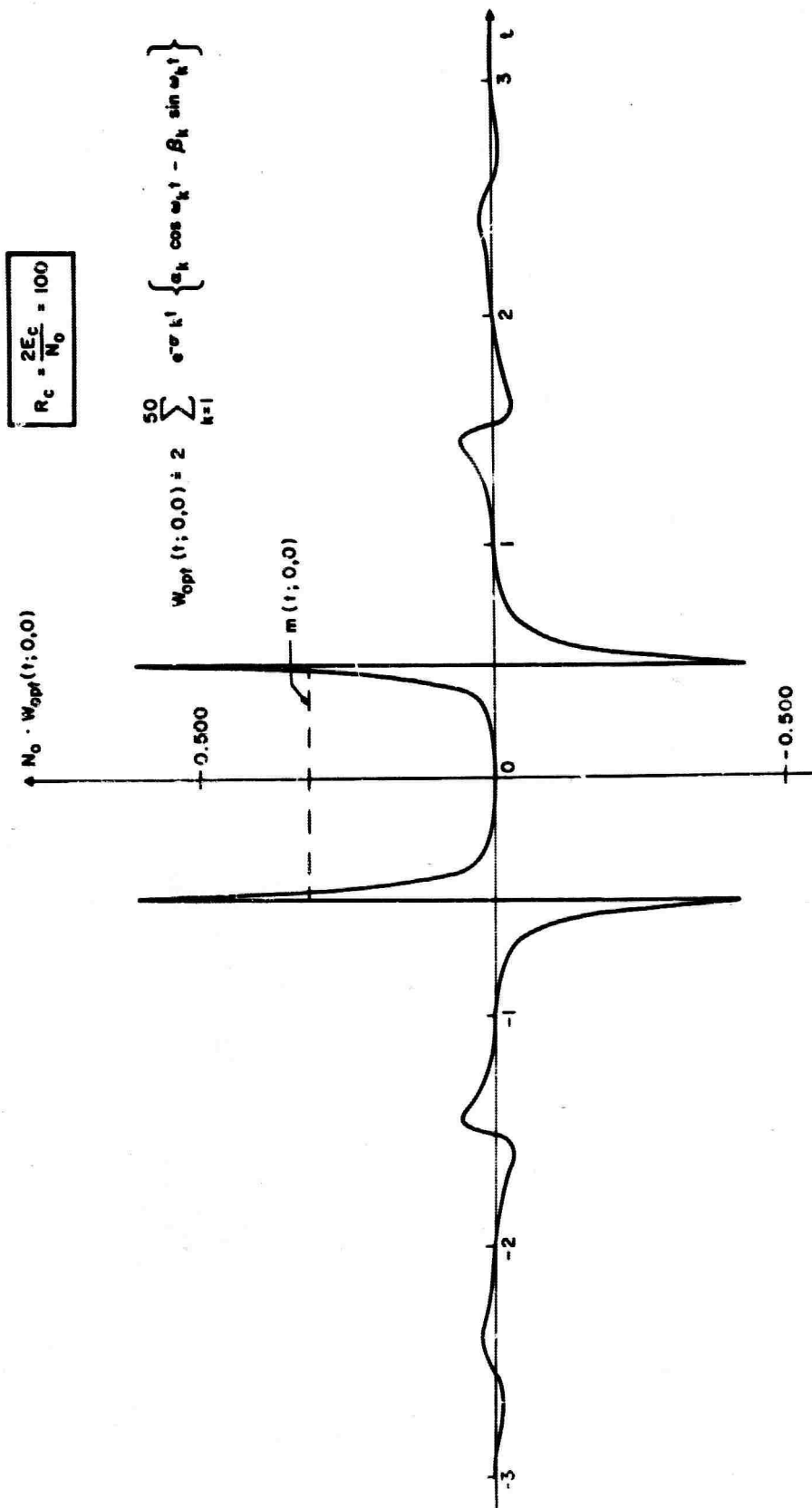


FIG. 24 OPTIMUM SMALL-NOISE PROCESSOR FOR A RECTANGULAR PULSE IN UNIFORMLY EXTENDED CLUTTER FROM A STATIONARY SOURCE

that the decay constant  $\sigma_k$  is directly proportional to the index  $k$  and, therefore, to the gross frequency of the  $k$ -th term of the solution. (See lqs. 12.67d and 12.67a.) For any fixed time  $t$ , therefore, the action of  $\exp\{-\sigma_k t\}$  is to attenuate by increasing factors the higher frequency components of the solution. This "filtering out" of higher frequency components contributes to the smoother appearance of Fig. 24 when compared to Fig. 23.

The small-noise solution shown in Fig. 24 was computed from the exact formula, Eq. (12.45), truncated after the first fifty terms. Exact† locations for the first fifty characteristic roots were used in computing the necessary residues  $a_{k2}$  according to Eqs. (12.39) through (12.41). These root locations were previously shown (in the first quadrant) in Fig. 22b. The many calculations which this procedure requires were performed with the aid of a digital computer.

#### D. SOLUTION FOR LARGE NOISE

The general Neumann series solution given by Eq. (7.5) provides a moderately convenient means for generating approximations to the optimum processor weighting function for the present case, in sufficiently large noise.

For clarity of exposition it is convenient to rewrite Eq. (7.5) in the form

$$w(t_1; \rho_o, f_o) = \frac{1}{N_o} m(t_1; \rho_o, f_o) + \frac{1}{N_o} \sum_{n=1}^{\infty} (-R_c)^n w_n(t_1, \rho_o, f_o) \quad (12.72)$$

---

† Exact, that is, to within tolerances of 0.01 applied to both real and imaginary components of the root location.

where

$$w_n(t_1; \rho_0, f_0) = \int_{-\infty}^{\infty} K_C^n(t_1, t_2) m(t_2; \rho_0, f_0) dt_2 \quad (12.73)$$

and where the iterated kernels  $K_C^n(t_1, t_2)$  are given by the original defining Eqs. (7.6) and (7.7). In the present case  $m(t_2; 0, 0)$  is the positive, rectangular pulse given by Eq. (12.4), or

$$m(t; 0, 0) = \begin{cases} T^{-\frac{1}{2}} & , \quad -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & , \quad \text{otherwise;} \end{cases} \quad (12.74a)$$

$$(12.74b)$$

while the normalized clutter dispersion function is given by

$$E(\rho, f) = 1 \cdot \delta(f) \quad (12.75)$$

Since the functions  $m(t; 0, 0)$  and  $E(\rho, f)$  are everywhere non-negative, one may readily verify by inspection of Eqs. (7.4), (7.6), (7.7), and (12.73), that the kernels  $K_C^n(t_1, t_2)$  and the functions  $w_n(t; 0, 0)$  are likewise non-negative for all values of their arguments, and for all  $n$ . From the considerations one concludes that, in the present case, the Neumann series solution of Eq. (12.72) actually gives  $w(t_1; 0, 0)$  in the form of an alternating series for any time  $t_1$ .

For the relatively simple waveforms of Eqs. (12.74) and (12.75), the functions  $K_C^n(t_1, t_2)$  and  $w_n(t_1; 0, 0)$  might, in principle, be generated by straightforward but exceedingly tedious integration. In fact, approximations to  $K_C^n(t_1, t_2)$

and  $w_n(t_1; 0, 0)$  were generated by numerical integration using a digital computer. The following bounds are abstracted from the numerical data which has been generated using  $T = 1$  in Eq. (12.74):

$$0 \leq w_1(t; 0, 0) \leq 0.750 \quad (12.76a)$$

$$0 \leq w_2(t; 0, 0) \leq 0.599 \quad (12.76b)$$

$$0 \leq w_3(t; 0, 0) \leq 0.511 \quad (12.76c)$$

$$0 \leq w_4(t; 0, 0) \leq 0.453 \quad (12.76d)$$

The result of evaluating the Neumann series of Eq. (12.72) for the values  $R_c = 0.2$  and  $0.4$ , while retaining only terms up to and including  $w_4(t; 0, 0)$ , is shown in Fig. 25.† For comparison purposes the optimum processor for  $R_c = 0$  is also indicated. Apart from the factor  $N_0$ , the latter processor is identical to the echo to be detected. It is the "matched" processor for the rectangular pulse.

The effects of clutter increasing from zero are clearly seen in Fig. 25. The "center" of the weight function becomes increasingly depressed, while increasing "undershoots" appear at  $t = \pm 0.5$ . It is not beyond credibility to say that the beginnings of the doublets which will eventually appear at  $t = \pm 0.5$  can be seen even for the small clutter levels which lead to Fig. 25.

From the progression of optimum processor weight functions which have now been seen in Figs. 23, 24, and 25, one

---

† The truncation error should at no point exceed about 1 per cent of the greatest value shown for  $w(t; 0, 0)$ , for  $R_c = 0.4$ .

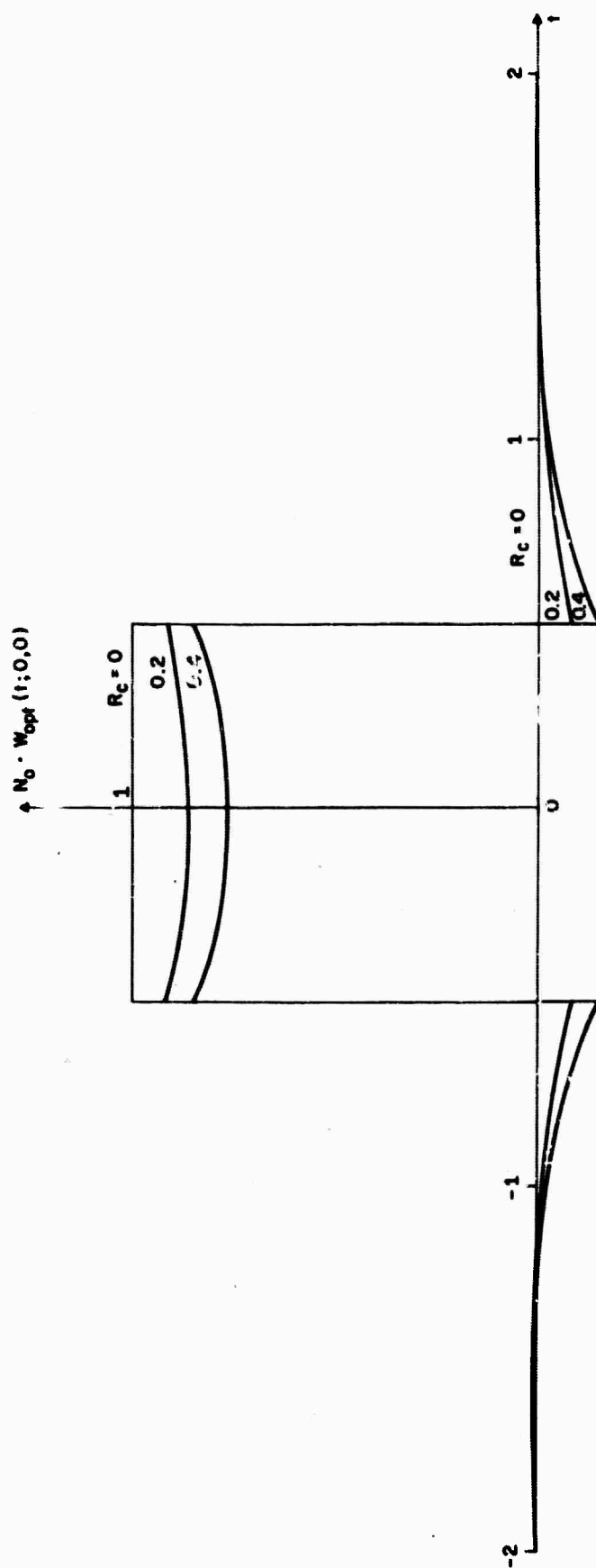


FIG. 25 OPTIMUM LARGE-NOISE PROCESSORS FOR A RECTANGULAR PULSE IN UNIFORMLY EXTENDED CLUTTER FROM A STATIONARY SOURCE

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

can hope to summarize the major characteristics of the optimum processor for detecting rectangular pulse echoes in clutter from a stationary source. When noise interference dominates, the optimum weight function resembles a matched processor (see Fig. 25). When clutter dominates, the processor includes doublets which have the effect of detecting the pulse by, in effect, detecting its edges. (See Fig. 23.) When both noise and clutter are present, the optimum weight function includes both of these characteristics in varying degrees, and definitely not in linear combination or superposition.

On the one hand, one sees that major features of the solution for noise-plus-clutter are reminiscent of the separate solutions for noise and clutter individually considered. On the other hand, one sees the complexity of a solution whose detailed structure depends in no obvious way upon the relative levels of noise and clutter interference.



XIII. DETECTION OF RECTANGULAR PULSES IN CLUTTER  
FROM A STATIONARY SOURCE OF FINITE EXTENT

This chapter will consider extensions of the results of the preceding chapter to cases involving more realistic clutter distributions.

It will be supposed that the clutter source, although still assumed stationary, may have a reflectivity (or radar cross-section) which varies with range delay. A dispersion function corresponding to this description has the form

$$\hat{G}(\rho, f) = \hat{G}_c \cdot E_c(\rho) \cdot \delta(f) \quad (13.1)$$

$$\text{where } \max_{\rho} E_c(\rho) = 1. \quad (13.2)$$

The modulation envelope  $m(t)$  is again assumed, as in equation (12.4), to be that of a rectangular pulse,

$$m(t) = \begin{cases} T^{-\frac{1}{2}}, & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0, & \text{otherwise.} \end{cases} \quad (13.3)$$

The linear operators  $L_t$ ,  $M_t$ , and  $D_t$ , corresponding to this pulse shape have already been given by equation (12.7). When the general equation (8.46) is applied to the present case, characterized by equations (13.1) through (13.3), the resulting equation to be solved is

$$\frac{2\hat{G}_c}{T} \cdot D_t E_c(t) D_t^+ \psi(t; \rho_o, f_o) - N_o \frac{d^2}{dt^2} \psi(t; \rho_o, f_o) = \frac{d}{dt} m(t; \rho_o, f_o) \quad (13.4a)$$

where the optimum weight function is to be found from

$$w(t; \rho_o, f_o) = - \frac{d}{dt} \psi(t; \rho_o, f_o) \quad (13.4b)$$

Equation (13.4a) is observed to be more complicated than its counterpart for uniform clutter, namely equation (12.8b), by exactly the presence of the variable coefficient  $E_c(t)$  which stems from equation (13.1). Consideration of a clutter source with spatially non-uniform reflectivity therefore elevates the problem to one of solving a difference-differential equation with variable, rather than constant, coefficients.

When the linear operator  $D_t E_c(t) D_t^+$  is expanded, and equation (13.4b) for  $w(t; \rho_o, f_o)$  is introduced into equation (13.4a), the result is

$$\begin{aligned} \frac{2E_c}{T} \left\{ E_c\left(t + \frac{T}{2}\right) w\left(t + T; \rho_o, f_o\right) - \left[ E_c\left(t + \frac{T}{2}\right) + E_c\left(t - \frac{T}{2}\right) \right] w(t; \rho_o, f_o) \right. \\ \left. + E_c\left(t - \frac{T}{2}\right) w\left(t - T; \rho_o, f_o\right) \right\} + N_o \frac{d^2}{dt^2} w(t; \rho_o, f_o) = \frac{d^2}{dt^2} m(t; \rho_o, f_o) \end{aligned} \quad (13.5)$$

which is rather more complicated than its earlier counterpart, equation (12.10).

A general, formal solution of equation (13.5) will not be attempted. Instead, certain characteristics of previous results in Chapter XII will be noted and used in a heuristic fashion to indicate the nature of exact or approximate solutions to equation (13.5) for various circumstances.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### A. SOURCE WITH UNIFORM CROSS SECTION

One simple clutter source model which fits into the present context is characterized by the dispersion function

$$\mathcal{E}(\rho, f) = \hat{\mathcal{E}}_c \cdot E_c(\rho) \cdot \delta(f) \quad (13.6a)$$

where

$$E_c(\rho) = \begin{cases} 1, & \frac{D}{2} \leq \rho \leq \frac{D}{2} \\ 0, & \text{otherwise} \end{cases} \quad (13.6b)$$

This corresponds to a clutter source which is strictly confined to a range-delay interval of  $D$  seconds, with constant reflectivity over that interval. No clutter energy originates outside of the interval, even through the received clutter interference will have a total duration exceeding  $D$ .

It may be verified that the introduction of equation (13.6b) for  $E_c(\rho)$  into equation (13.5) will have the effect of replacing equation (13.5) by a set of difference-differential equations, each with constant coefficients. Each equation of the set will characterize the solution  $w(t; \rho_o, f_o)$  over a different interval of the  $t$ -axis. By suitable adjustment of the homogeneous solutions of each equation of the set one would expect to find the total solution for  $w(t; \rho_o, f_o)$  on the entire  $t$ -axis. The solution for  $w(t; \rho_o, f_o)$  would be in the form of a weighted sum of particular and homogeneous solutions of each equation of the set and could, in principle, be derived for arbitrary clutter and noise levels.

That this rather formidable procedure can be circumvented when noise is neglected will now be seen.

1. A Set of Reciprocal Waveforms

Consider the set of waveforms  $m_k(t)$  defined by

$$m_k = m(t - k\tau) \quad k = 0, \pm 1, \pm 2, \dots \quad (13.7)$$

where 
$$m(t) = \begin{cases} T^{-\frac{1}{2}} & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \quad (13.8)$$

The waveforms  $m_k(t)$  are therefore replicas of the rectangular pulse  $m(t)$  originally given by equation (13.3), each delayed (or advanced) by a multiple of  $\tau$  seconds. It will be supposed that  $\frac{T}{\tau}$  is an integer greater than unity, with the result that typical waveforms  $m_k(t)$  will appear as in Figure 26.

The question of interest is: "What are the waveforms  $\omega_k(t)$  which are reciprocal to the  $m_k(t)$ ?" That is, find the waveforms  $\omega_k(t)$  which have the properties

$$\int_{-\infty}^{\infty} \omega_k(t) m_k(t) dt = 1 \quad (13.9a)$$

and 
$$\int_{-\infty}^{\infty} \omega_k(t) m_j(t) dt = 0 \quad \text{for } j \neq k. \quad (13.9b)$$

An answer<sup>†</sup> to the question is given in the following manner.

Consider an elemental waveform  $\omega(t)$  with the properties

$$1) \quad \omega(t) = 0, \quad \text{if } |t| > \frac{\tau}{2} \quad (13.10a)$$

<sup>†</sup> It will soon be seen that there are many sets of waveforms with the desired properties.

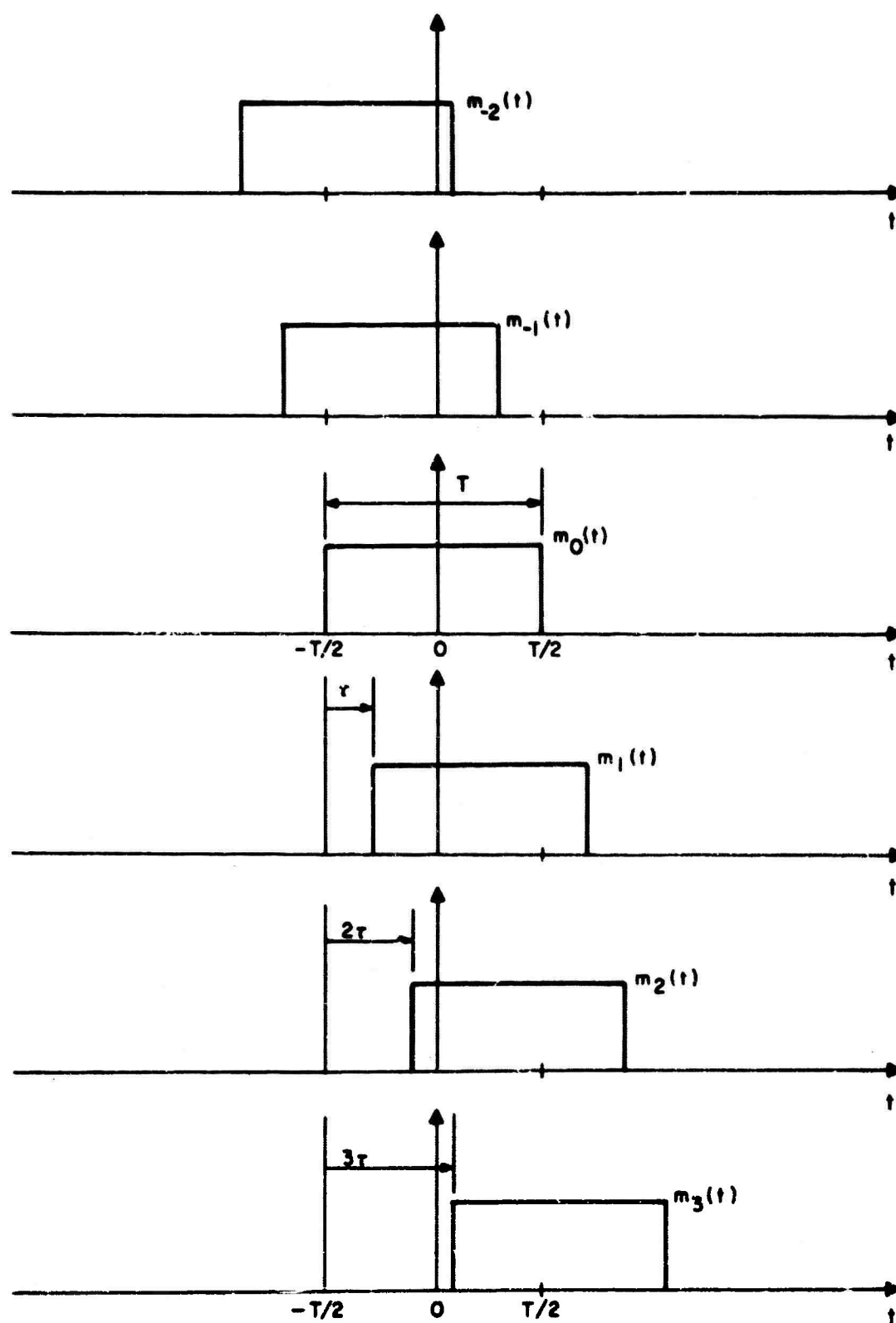


FIG. 26 A SET OF WAVEFORMS

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$ii) \int_{-\infty}^0 \omega(t) dt = + \frac{1}{2} T^{\frac{1}{2}} \quad (13.10b)$$

$$iii) \int_0^{\infty} \omega(t) dt = - \frac{1}{2} T^{\frac{1}{2}} \quad (13.10c)$$

A typical, but certainly not necessary, example of such a waveform is

$$\omega(t) = \begin{cases} -\frac{\pi^{\frac{1}{2}}}{2\tau} \sin \frac{2\pi t}{\tau} & |t| < \frac{\tau}{2} \quad (13.11a) \\ 0 & |t| > \frac{\tau}{2} \quad (13.11b) \end{cases}$$

Consider next the waveform  $\omega_k(t)$  defined, for all integral  $k$ , by

$$\omega_k(t) = \sum_{l=0}^{\infty} \omega(t - k\tau - [l + \frac{1}{2}] T) \quad \text{for } t \geq k\tau \quad (13.12a)$$

$$\text{and } \omega_k(t) = \omega_k(2k\tau - t) \quad \text{for } t < k\tau. \quad (13.12b)$$

Such a waveform  $\omega_k(t)$ , based on the particular  $\omega(t)$  defined in equation (13.11), is shown in Figure 27 together with several  $m_j(t)$  waveforms.

The assertion is that the set of  $\omega_k(t)$  defined by equations (13.10) and (13.12) constitute a waveform set which is reciprocal to the  $m_k(t)$ . This may be verified algebraically by using equations (13.10) and (13.12) to show that the  $\omega_k(t)$  so defined also have the necessary properties (13.9a) and (13.9b). However, it is more easily verified, and with greater insight, by considering Figure 27.

Figure 27 has been drawn for the parameter ratio  $\frac{T}{\tau}$  equal to 5. The top graph is the (single) waveform

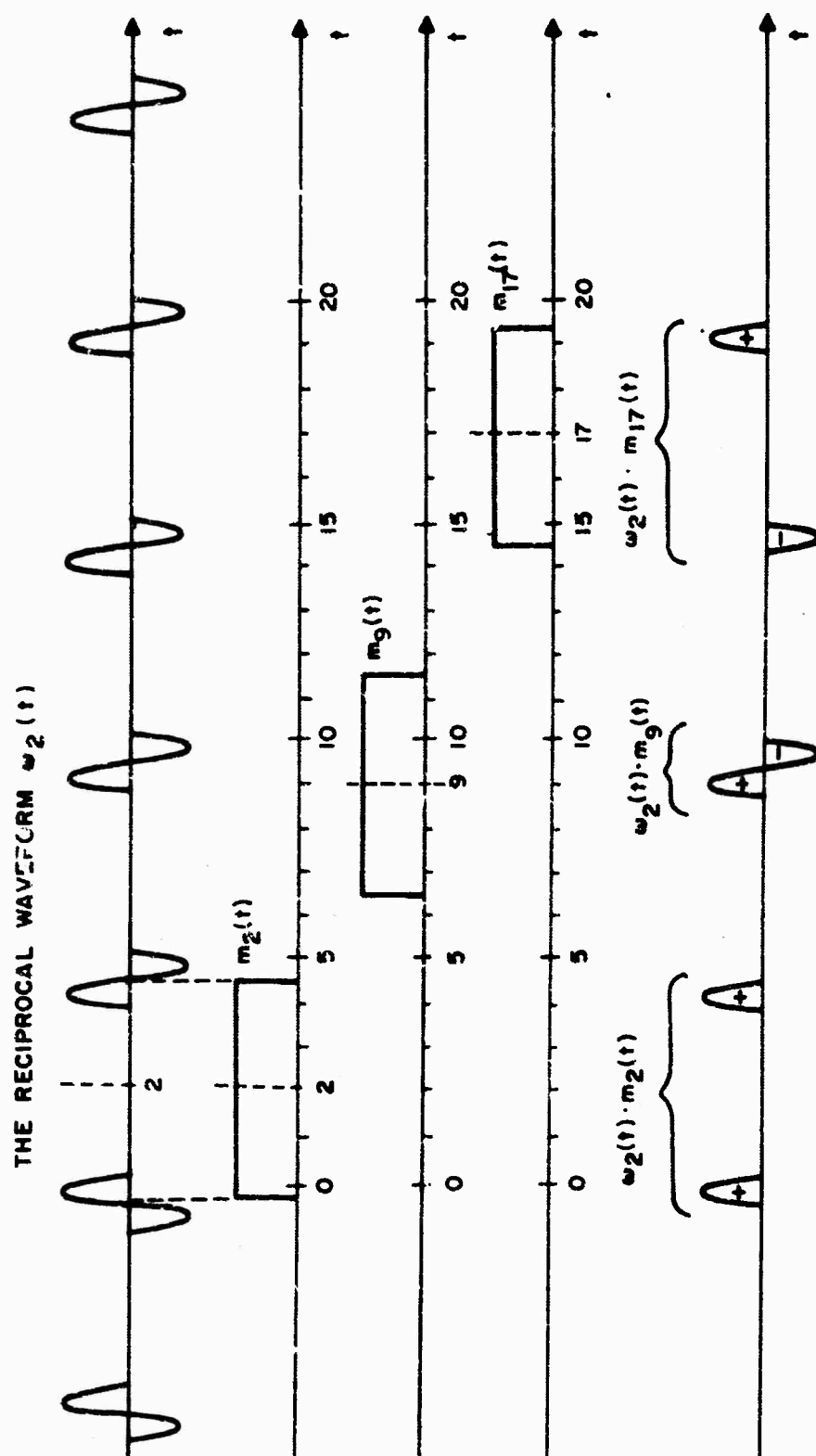


FIG. 27 A RECIPROCAL WAVEFORM

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$\omega_2(t)$ , while the next line of graphs comprises  $m_2(t)$ ,  $m_9(t)$ , and  $m_{17}(t)$ . The lowest set of graphs contains the products of  $\omega_2(t)$  with each of the preceding  $m_k(t)$ , and inspection reveals that

$$\int_{-\infty}^{\infty} \omega_2(t) m_2(t) dt \neq 0 \quad (13.13a)$$

while 
$$\int_{-\infty}^{\infty} \omega_2(t) m_9(t) dt = 0 \quad (13.13b)$$

and 
$$\int_{-\infty}^{\infty} \omega_2(t) m_{17}(t) dt = 0 \quad (13.13c)$$

Since these are seen to correspond to the three types of situation which can arise between  $m_k(t)$  and  $\omega_j(t)$ , it should be clear that the waveforms  $\omega_k(t)$  are indeed reciprocal to the  $m_k(t)$ .†

Figure 27 also makes clear that the detailed shape of the elemental function  $\omega(t)$  is not of great concern, as long as its area is disposed in accord with equations (13.10).

Finally, the purpose of introducing the elemental function  $\omega(t)$ , with its particular properties, is for its resemblance to the impulse doublets which appeared in Sec. XII. To make the resemblance clearer, one may define the function

$$\Omega(t) = \begin{cases} \frac{1}{\tau} \left[ 1 + \cos \frac{2\pi t}{\tau} \right], & |t| \leq \frac{\tau}{2} \\ 0, & \text{otherwise,} \end{cases} \quad (13.14a)$$

$$(13.14b)$$

† That the  $\omega_k(t)$  have the proper amplitude for unit projection upon the appropriate  $m_k(t)$  may be verified algebraically.



and verify that

$$\int_{-\infty}^{\infty} \Omega(t) dt = 1. \quad (13.15)$$

Because of these properties,  $\Omega(t)$  may be regarded as an approximation to the unit impulse function  $\delta(t)$ . Moreover, from equations (13.14) and (13.11) one can now write

$$\omega(t) = \frac{\tau}{4} \cdot T^{\frac{1}{2}} \cdot \frac{d}{dt} \Omega(t). \quad (13.16)$$

The resemblance between the reciprocal function  $\omega_k(t)$  depicted in Figure 27, and characterized by equations (13.12) and (13.16), and the zero-noise optimal solution  $w_{opt}(t;0,0)$  depicted in Figure 23 and given by equations (12.71) of the preceding chapter, should now be evident.

## 2. Optimum Processor (No Noise)

The discussion of the preceding chapter suggests the form of solution for the present problem, where  $E_c(\rho)$  is given by equation (13.6) and repeated here

$$E_c(\rho) = \begin{cases} 1, & \frac{D}{2} \leq \rho \leq \frac{D}{2} \\ 0, & \text{otherwise.} \end{cases} \quad (13.17)$$

Let us for the moment restrict attention to the problem of detecting the rectangular pulse  $m(t;0,0)$ , given by equation (13.3), in the clutter interference (without noise) characterized by equation (13.17).

For  $D = \infty$  in equation (13.17), the solution has already been given in the preceding chapter by equations (12.71), which are repeated here,

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$w(t;0,0) = \frac{1}{2\hat{\epsilon}_c} \sum_{k=0}^{\frac{1}{2}\infty} \frac{d}{dt} \delta(t - \frac{T}{2} - kT) \quad t > 0 \quad (13.18a)$$

$$\text{and } w(t;0,0) = w(-t;0,0) \quad t < 0. \quad (13.18b)$$

One may now either follow the heuristic reasoning of the preceding section, or use equation (8.16), to demonstrate that

$$\int_{-\infty}^{\infty} w^*(t;0,0) m(t;p,0) dt = 0 \quad \text{for } p \neq 0 \quad (13.19)$$

However, the clutter resulting from the dispersion function of equation (13.17) is just exactly the superposition of waveforms  $m(t;p,0)$ , with  $-\frac{D}{2} \leq p \leq \frac{D}{2}$ , essentially according to <sup>†</sup>

$$c(t) = \int_{-D/2}^{D/2} a(p) \cdot m(t;p,0) dp \quad (13.20)$$

$$\text{where } \langle |a(p)|^2 \rangle = 2\hat{\epsilon}_c. \quad (13.21)$$

In view of equations (13.19) through (13.21), one suspects that the solution for the present case will be as shown in Figure 28. It is identical to  $w(t;0,0)$  given in equations (13.18) for times  $t$  where the clutter interference exists. It is taken to be zero where clutter does not exist, and may be written

<sup>†</sup> See equations (4.7), (4.8), and (4.11) for the origins of this simplified representation.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

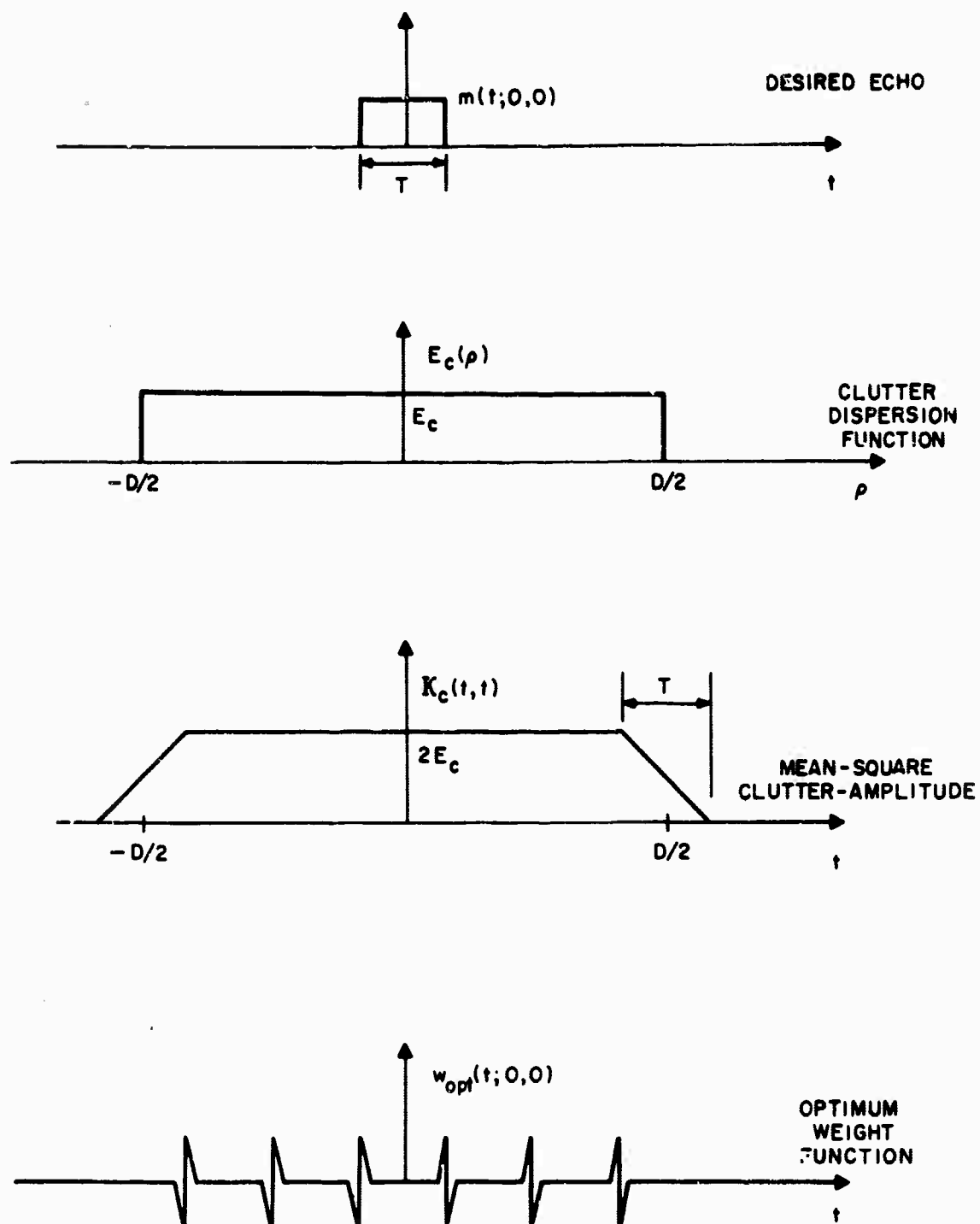


FIG. 28 OPTIMUM ZERO-NOISE PROCESSOR FOR A RECTANGULAR PULSE  
IN UNIFORM CLUTTER OF FINITE EXTENT

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$w_{\text{opt}}(t;0,0) = \frac{1}{2} \frac{T^{\frac{1}{2}}}{c} \sum_{k=0}^{\infty} \frac{d}{dt} \delta(t - \frac{T}{2} - kT) \quad 0 < t < \frac{1}{2}(D + T) \quad (13.22a)$$

$$w_{\text{opt}}(t;0,0) = w_{\text{opt}}(-t;0,0) \quad -\frac{1}{2}(D + T) < t < 0 \quad (13.22b)$$

$$\text{and } w_{\text{opt}}(t;0,0) = 0 \quad \text{otherwise} \quad (13.22c)$$

It may be verified, although details will not be given here, that  $w_{\text{opt}}(t;0,0)$  given in the preceding equations does indeed satisfy equation (13.5) for an optimum processor (with  $N_c = 0$ ).

Following the example set in Figure 28, one may derive the various (zero-noise) optimum processors shown in Figure 29 for echoes with various positions within the clutter interference. In all cases the optimum<sup>†</sup> processors have the following common characteristics:

- (i) the relative times of occurrence of the impulse doublets are determined by the leading and trailing edges of the echo to be detected;
- (ii) except for truncation effects, the weight function is symmetric about the time of occurrence of the echo; and
- (iii) the weight function is non-zero only over the interval where clutter is present.

---

<sup>†</sup> The processors shown in Figure 29 do satisfy equation (13.5) and are, therefore, optimum.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

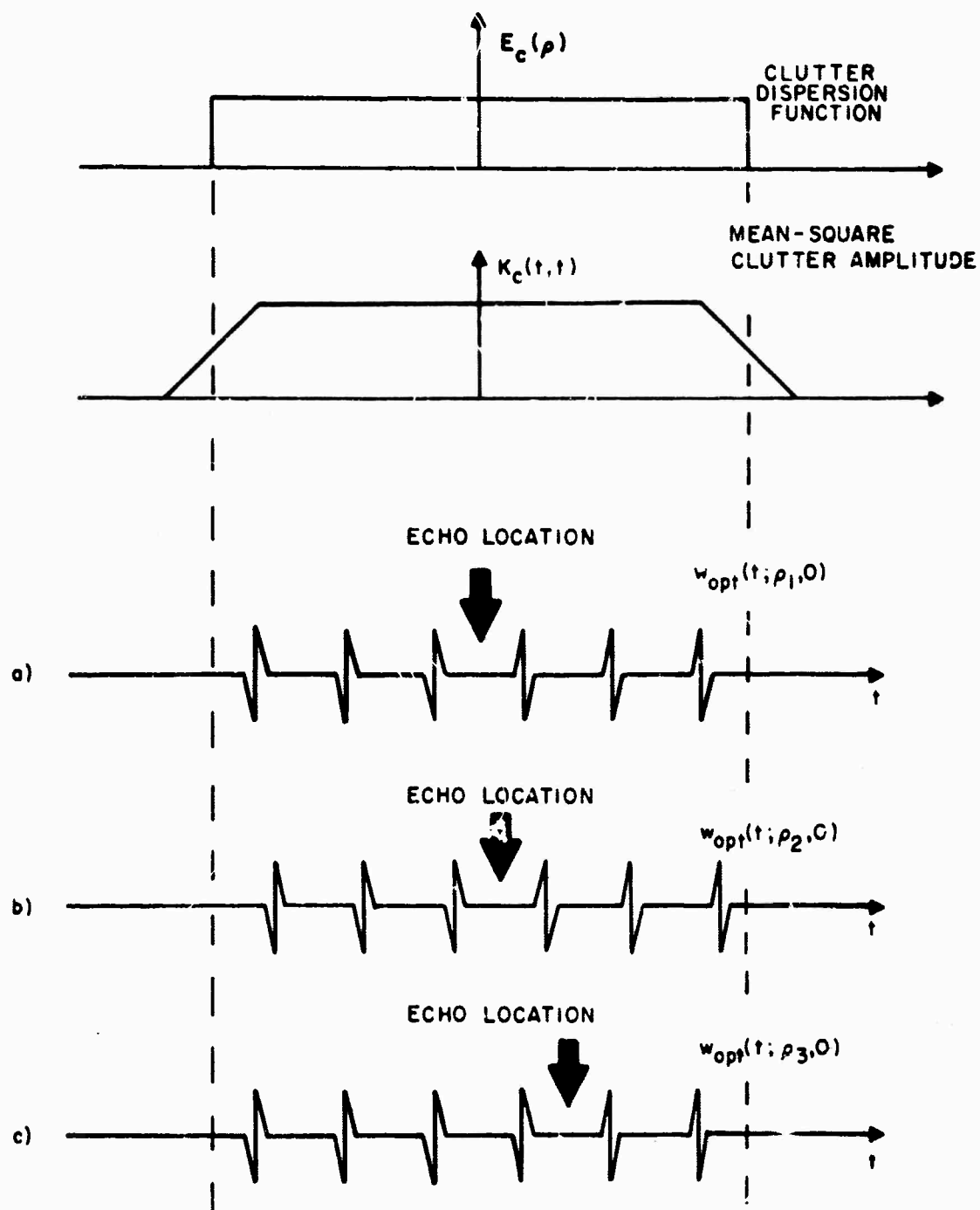


FIG. 29 OPTIMUM ZERO-NOISE PROCESSORS FOR VARIOUS PULSE-LOCATIONS IN UNIFORM CLUTTER OF FINITE EXTENT

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

It is evident from Figure 29 that these common characteristics lead to weight functions which can have marked differences, depending upon the echo location. Thus, for example, a small retardation of the echo occurrence as shown between Figures 29a and 29b, leads to a corresponding retardation of the weight function. The further retardation which leads to Figure 29c, however, causes the suppression of one doublet from the trailing end weight function and the addition of a doublet of opposite "sign" at the leading end.

The weight functions shown in Figures 29b and 29c are seen to have essentially different structures. The optimum (no-noise) solution to the present problem is, therefore, properly called a time-varying processor, since time-translation of the input echo does not result in only time-translation of the corresponding optimum processor.

A formal expression for the optimum processors presented in this section, and in Figure 29, may also be written. It is:

$$w_{\text{opt}}(t; \rho_0, 0) = \frac{1}{2\epsilon_c} \sum_{k=-\infty}^{\infty} \text{sgn}(t - \rho_0) \cdot \frac{d}{dt} \delta(t - \rho_0 - \frac{T}{2} - kt) \quad (13.23a)$$

for  $-\frac{1}{2}(D + T) < t < \frac{1}{2}(D + T)$ , and

$$w_{\text{opt}}(t; \rho_0, 0) = 0, \text{ otherwise} \quad (13.23b)$$

### B. SOURCE WITH NON-UNIFORM CROSS SECTION

Attention will now be returned to the more general dispersion function originally given in equation (13.1), namely

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$\mathcal{E}(\rho, f) = \hat{\mathcal{E}}_c \cdot E_c(\rho) \cdot \delta(f) \quad (13.24)$$

Presentation of the zero-noise optimum processor for this more general dispersion function is now a simpler matter because solutions in less general cases have been seen in preceding sections.

Let it be supposed that the optimum weight function for detecting  $m(t;0,0)$  in clutter characterized by equation (13.24), has the form

$$w(t;0,0) = \sum_{k=-\infty}^{\infty} a_k \delta^{(1)}\left(t - \frac{T}{2} - kT\right) \quad (13.25)$$

where the coefficients  $a_k$  are to be determined, and where

$$\delta^{(1)}(t) \equiv \frac{d}{dt} \delta(t) \quad (13.26)$$

The form of equation (13.25) is suggested by the form of the previous solution, equation (12.71), for uniformly extended stationary clutter. In proposing equation (13.25) it is tentatively assumed that inclusion of the unspecified coefficients  $a_k$  will be sufficient to permit a solution for the non-uniform clutter of the present case. This remains to be demonstrated.

The basic integral equation which must be satisfied by  $w(t;0,0)$  of equation (13.25) is

$$\int_{-\infty}^{\infty} \chi_c(t_1, t_2) w(t_2;0,0) dt_2 = m(t_1;0,0) \quad (13.27)$$

This is the same as equation (6.2) with  $N_0, \rho_0$ , and  $f_0$  set equal to zero. The covariance kernel which results when equation (13.24) is used in the defining equation (6.3) is

$$\chi_c(t_1, t_2) = 2\hat{\mathcal{E}}_c \int_{-\infty}^{\infty} E_c(\rho) m(t_1 - \rho) m^*(t_2 - \rho) d\rho \quad (13.28)$$

When equations (13.27) and (13.28) are combined, the result is

$$2\hat{\mathcal{E}}_c \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_c(\rho) m(t_1 - \rho) m^*(t_2 - \rho) w(t_2; 0, 0) dt_2 d\rho = m(t_1; 0, 0) \quad (13.29)$$

The coefficients  $a_k$  in equation (13.25) will now be found by requiring that  $w(t; 0, 0)$  of equation (13.25) be a solution of equation (13.29).

The integration with respect to  $t_2$  in equation (13.29) may be performed with the aid of equations (8.16) and (13.25). The result is

$$\int_{-\infty}^{\infty} m^*(t_2 - \rho) w(t_2; 0, 0) dt_2 = \sum_{k=-\infty}^{\infty} a_k (-1) \left[ \frac{d}{dt} m^*(t - \rho) \right]_{t = \frac{T}{2} + kt} \quad (13.30)$$

or, if equation (12.5) is used for the derivative of  $m(t)$ ,

$$\int_{-\infty}^{\infty} m^*(t_2 - \rho) w(t_2; 0, 0) dt_2 = \frac{1}{T^{\frac{1}{2}}} \sum_{k=-\infty}^{\infty} (-1) a_k \left[ \delta(T + kT - \rho) - \delta(kT - \rho) \right] \quad (13.31)$$

When this expression is introduced into equation (13.29), the integration with respect to  $\rho$  may be performed. The result is

$$\frac{2\hat{\mathcal{E}}_c}{T^{\frac{1}{2}}} \cdot \sum_{k=-\infty}^{\infty} (-1) a_k \left[ E_c(T + kT) m(t_1 - T - kT) - E_c(kT) m(t_1 - kT) \right] = m(t_1) \quad (13.32)$$



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

The coefficients of concurring waveforms on each side of this equation may now be identified. One concludes that for equation (13.32) to be an identity, the coefficients  $a_k$  must be related by

$$\frac{2\hat{G}_c}{T^{\frac{1}{2}}} (-1) \left[ a_{-1} \cdot E_c(0) - a_0 \cdot E_c(0) \right] = 1 \quad (13.33a)$$

while

$$\frac{2\hat{G}_c}{T^{\frac{1}{2}}} \left[ a_{k-1} \cdot E_c(kT) - a_k \cdot E_c(kT) \right] = 0, \quad k \neq 0. \quad (13.33b)$$

From the latter of these two relations one concludes that

$$a_k = a_0 \quad k = 1, 2, 3, \dots, \quad (13.34a)$$

and that

$$a_k = a_{-1} \quad k = -2, -3, -4, \dots \quad (13.34b)$$

Thus all coefficients  $a_k$  are determined once the two coefficients  $a_0$  and  $a_{-1}$  have been given values. The latter two, in turn, are constrained to satisfy equation (13.33a) or

$$a_0 - a_{-1} = \frac{T^{\frac{1}{2}}}{2\hat{G}_c \cdot E_c(0)} \quad (13.35)$$

The result is that the optimum processor for this case can be written

$$w(t;0,0) = \begin{cases} a_0 \sum_{k=0}^{\infty} \delta^{(1)}(t - \frac{T}{2} - kT) & t > 0 \\ a_{-1} \cdot \sum_{k=-\infty}^{-1} \delta^{(1)}(t - \frac{T}{2} - kT) & t < 0 \end{cases} \quad (13.36a) \quad (13.36b)$$

where the only remaining task is to find the correct amplitude  $a_0$ . The amplitude  $a_{-1}$  will then also have been determined, by equation (13.35).

It is at this point that a certain formal indeterminacy may be seen in the solution of equation (13.27) for the present case. Note first that if the "function"  $w_0(t)$  is defined by, for all  $t$ ,

$$w_0(t) = \sum_{k=-\infty}^{\infty} \delta^{(1)}(t - \frac{T}{2} - kT) \quad (13.37)$$

then it may be verified (using equation 13.31) that

$$\int_{-\infty}^{\infty} \mathcal{H}_c(t_1, t_2) w_0(t_2) dt_2 = 0. \quad (13.38)$$

Thus  $w_0(t)$  provides what might be termed the "homogeneous" solution to equation (13.27) for the present case of a non-uniform, stationary clutter source.

Observe next that if one determines a solution  $w_1(t;0,0)$  according to equations (13.36) for a particular  $a_0 = a$ , and then determines a second solution  $w_2(t;0,0)$  for  $a_0 = a + \alpha$ , then the two solutions will be related by

$$w_2(t;0,0) = w_1(t;0,0) + \alpha \cdot w_0(t). \quad (13.39)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

That is to say, variations of the coefficient  $a_0$  appearing in equations (13.36) lead only to variations in the amplitude of the homogeneous component of the solution defined by equations (13.36).

That a homogeneous solution to the basic integral equation (13.27) exists in this case therefore permits the existence of a family of solutions interrelated by an equation such as (13.39).

In these circumstances, the single solution which will be called optimal in this research is that corresponding to the choice of coefficients

$$a_0 = -a_{-1} = \frac{T^{\frac{1}{2}}}{4\mathcal{E}_c(0)}. \quad (13.40)$$

The corresponding weight function is given by, for  $t \geq 0$ ,

$$w_{\text{opt}}(t;0,0) = \frac{T^{\frac{1}{2}}}{4\mathcal{E}_c(0)} \sum_{k=0}^{\infty} \frac{d}{dt} \delta\left(t - \frac{T}{2} - kT\right) \quad (13.41a)$$

and, for  $t < 0$ ,

$$w_{\text{opt}}(t;0,0) = -w_{\text{opt}}(-t;0,0). \quad (13.41b)$$

It may be verified that  $w_{\text{opt}}(t;0,0)$  thus defined has no homogeneous component. That is,  $w_{\text{opt}}(t;0,0)$  cannot be represented in the manner of equation (13.39) with any non-zero value for  $\alpha$ .

It may also be verified that  $w_{\text{opt}}(t;0,0)$  given by equation (13.41) is identical to the processor given earlier by equations (12.71). This is a more striking conclusion,

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

which asserts that the processor derived earlier for the case of a uniformly extended stationary clutter source (equation 12.71) is also the optimum processor for the present case of a non-uniformly extended stationary clutter source.

The fact is illustrated in Figure 30, where the processor weight function for the signal  $m(t;0,0)$  is identical to that given earlier in Figure 23, even though the mean-square clutter amplitude is now a function of range delay.

It must finally be remarked that this solution, although strange at first glance, is nevertheless entirely consonant with previous remarks concerning reciprocal waveforms.

In particular, one may refer to the reciprocal waveform  $\omega_2(t)$  shown in Figure 23 and described by equations (13.10) through (13.13). In connection with Figure 23 it was shown that  $\omega_2(t)$  was orthogonal, for example, to  $m_{17}(t)$ . This was stated in equation (13.13c) as

$$\int_{-\infty}^{\infty} \omega_2(t) m_{17}(t) dt = 0 \quad (13.42)$$

In the present context it is relevant to observe that this orthogonality persists irrespective of the actual amplitude of  $m_{17}(t)$ .

In the detection of  $m_2(t)$ , for which  $\omega_2(t)$  is an optimal processor, the waveform  $m_{17}(t)$  may be regarded as a typical clutter component. The fact observed earlier was that  $\omega_2(t)$  was orthogonal to all clutter components not identical to the desired echo. In the present context one notes that this orthogonality remains true even though the different clutter components may have different (mean-square) amplitudes. One therefore might have expected that

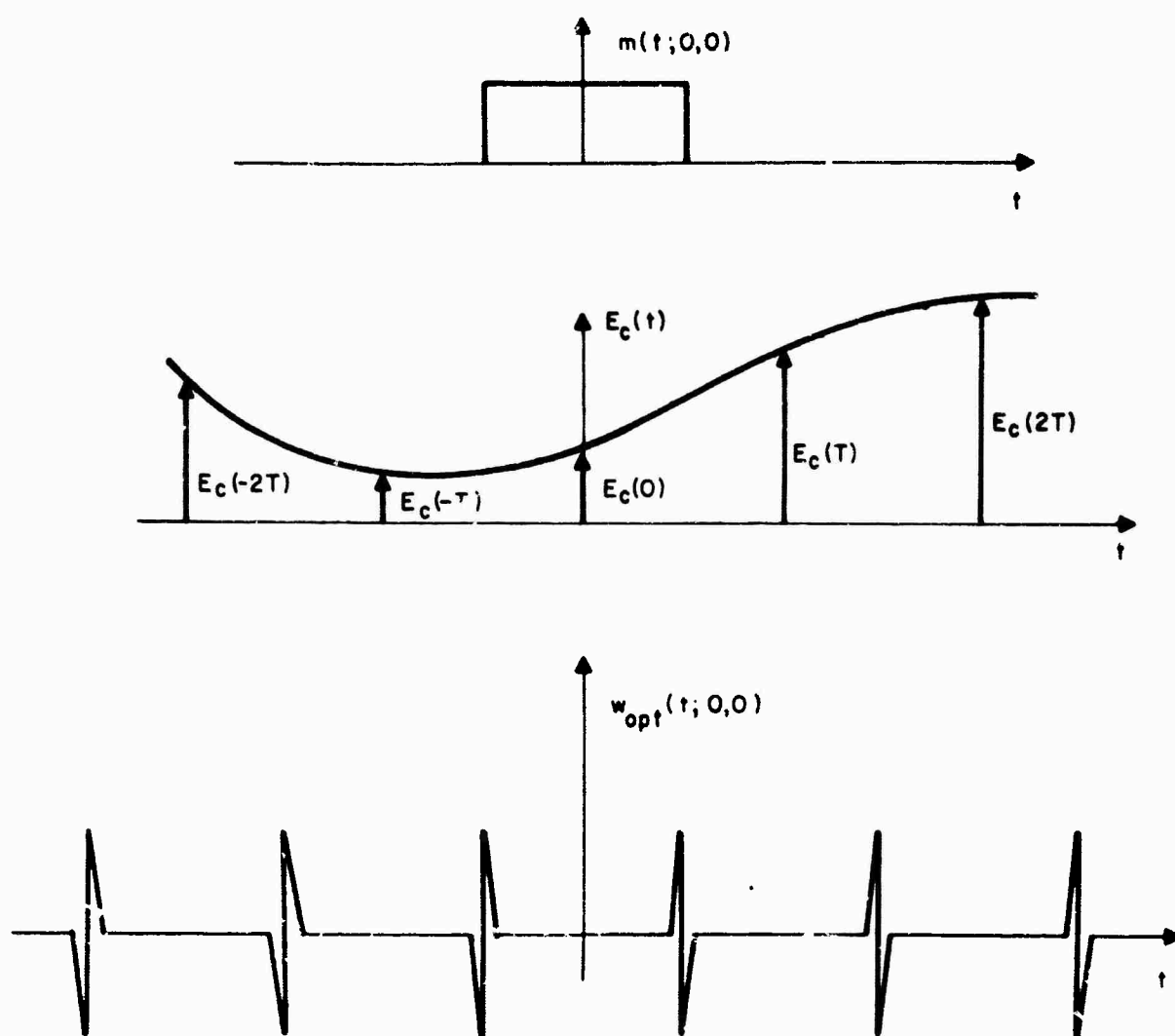


FIG. 30 OPTIMUM ZERO-NOISE PROCESSOR FOR A RECTANGULAR PULSE  
IN NON-UNIFORM CLUTTER

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

the reciprocal waveform which arose in the case of uniform clutter (equation 12.71) would be the same reciprocal waveform for optimum detection in non-uniform clutter, as was indeed shown to be the case at equation (13.41).

APPENDIX AFORMAL EXPRESSIONS FOR AN OPTIMAL SYSTEM

A weight function  $w(t)$  is to be found which maximizes a ratio of the form

$$\frac{S}{I} = \frac{\left| \int_{-\infty}^{\infty} w^*(t)x(t)dt \right|^2}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1)w(t_2)K(t_1, t_2)dt_1 dt_2} \quad (A-1)$$

This may be regarded<sup>51</sup> as a problem in the calculus of variations with the object of minimizing the denominator subject to the constraint of the numerator being constant.

The numerator may be rewritten as a double integral and the problem converted to one of unconstrained minimization by the introduction of a Lagrange multiplier  $\lambda$ . Then a condition for a solution is that the first variation of  $F$  be zero, where

$$F = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1)w(t_2)K(t_1, t_2)dt_1 dt_2 + \lambda \left\{ |K|^2 - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1)w(t_2)x(t_1)x^*(t_2)dt_1 dt_2 \right\} \quad (A-2)$$

and  $|K|^2$  = the constrained value of the numerator.

Upon replacing  $w(t)$  by  $w(t) + \delta w(t)$ , where  $\delta w(t)$  is an arbitrary variation of  $w(t)$ , one finds that the first variation of  $F$  is given by

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$\begin{aligned} \delta F = & \int_{-\infty}^{\infty} \delta w^*(t_1) \int_{-\infty}^{\infty} w(t_2) \{K(t_1, t_2) - \lambda x(t_1) x^*(t_2)\} dt_1 dt_2 \\ & + \int_{-\infty}^{\infty} \delta w(t_2) \int_{-\infty}^{\infty} w^*(t_1) \{K(t_1, t_2) - \lambda x(t_1) x^*(t_2)\} dt_1 dt_2 . \end{aligned} \quad (A-3)$$

In the present problem the kernel  $K(t_1, t_2)$  is Hermitian. That is

$$K(t_1, t_2) \equiv K^*(t_2, t_1) . \quad (A-4)$$

In this circumstance one may verify that the inner integral in the first term of (A-3) is the complex conjugate of the inner integral in the second term. Since the arbitrary variation  $\delta w(t)$  can arbitrarily be real or imaginary, a necessary condition for

$$\delta F = 0 \quad (A-5)$$

is that

$$\int_{-\infty}^{\infty} w(t_2) \{K(t_1, t_2) - \lambda x(t_1) x^*(t_2)\} dt_2 = 0 . \quad (A-6)$$

Both the real and imaginary parts of this generally complex integral must be zero.

The necessary condition which  $w(t)$  must satisfy is derived by rewriting (A-6) to read

$$\int_{-\infty}^{\infty} K(t_1, t_2) w(t_2) dt_2 = x(t_1) \cdot \lambda \int_{-\infty}^{\infty} w(t_2) x^*(t_2) dt_2 \quad (A-7)$$

and recalling the numerator constraint, which may be written

$$\int_{-\infty}^{\infty} w(t_2) x^*(t_2) dt_2 = K . \quad (A-8)$$



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

From (A-7) and (A-8) one concludes that

$$\int_{-\infty}^{\infty} K(t_1, t_2) w(t_2) dt_2 = x(t_1) \cdot u \quad (A-9)$$

where  $u$  is a possibly complex constant. Since the ratio  $\frac{S}{I}$  is independent of amplitude scale changes in  $w(t)$ , the factor  $u$  may be taken equal to unity without essentially affecting results.

Equation (A-9) expresses  $x(t)$  as a linear transformation of the unknown function  $w(t)$ . A formal solution to (A-9) may be derived by assuming that the function  $w(t)$  can be expressed conversely as a linear transformation of  $x(t)$ . That is, a kernel  $\mathcal{L}(t_1, t_2)$  will be assumed to exist such that

$$w(t_1) = \int_{-\infty}^{\infty} \mathcal{L}(t_1, t_2) x(t_2) dt_2. \quad (A-10)$$

When (A-10) is introduced in (A-9) the result may be written (with  $u=1$ )

$$\int_{-\infty}^{\infty} \sigma(t_1, t_3) x(t_3) dt_3 \equiv x(t_1) \quad (A-11)$$

where

$$\sigma(t_1, t_3) = \int_{-\infty}^{\infty} K(t_1, t_2) \mathcal{L}(t_2, t_3) dt_2. \quad (A-12)$$

Equation (A-11) asserts that the kernel  $\sigma(t_1, t_3)$  defined by (A-12) has a certain "sifting" property with respect to the function  $x(t)$ . The conclusion is that, if a kernel  $\mathcal{L}(t_2, t_3)$  exists satisfying (A-12) then  $w(t)$  given by (A-10) will be a solution of the desired Eq. (A-9). Equations (A-10) through (A-12) therefore describe a formal solution to the problem of maximizing the ratio  $\frac{S}{I}$ .

A formal expression for the maximum value attained by the ratio can also be derived. For the numerator in (A-1) one

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

computes

$$\int_{-\infty}^{\infty} w(t_1) x^*(t_1) dt_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^*(t_1) \mathcal{L}(t_1, t_2) x(t_2) dt_1 dt_2. \quad (\text{A-13})$$

Equation (A-10) is also used to compute, for the denominator of (A-1),

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_1) \mathcal{K}(t_1, t_2) w(t_2) dt_1 dt_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{L}^*(t_1, t_2) x^*(t_2) \mathcal{K}(t_1, t_3) \mathcal{L}(t_3, t_4) x(t_4) dt_1 dt_2 dt_3 dt_4 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{L}^*(t_1, t_2) x^*(t_2) \sigma(t_1, t_4) x(t_4) dt_1 dt_2 dt_4 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t_1) \mathcal{L}^*(t_1, t_2) x^*(t_2) dt_1 dt_2 \quad (\text{A-14}) \end{aligned}$$

wherein the sifting property of  $\sigma(t_1, t_4)$  has been used.

Since the right-hand sides of (A-13) and (A-14) are complex conjugates of each other, their substitution into (A-1) yields

$$\frac{S}{I} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^*(t_1) \mathcal{L}(t_1, t_2) x(t_2) dt_1 dt_2 \quad (\text{A-15})$$

for the maximum value of  $\frac{S}{I}$ .

APPENDIX BANALYSIS IN THE FREQUENCY DOMAIN

Presented in this appendix is a selection of identities which relate expressions given in terms of time functions to expressions which involve their Fourier transforms. The purpose is to exhibit those relations which facilitate the analysis in the text.

The first section contains certain results from Fourier transform theory. The following sections sketch the derivations of various results cited in the text.

1. Excerpts from Fourier Transform Theory

Time functions will be denoted by lower case letters and covariance functions by script letters. Their frequency transforms will be denoted by corresponding capital letters. A given function  $g(t)$  and its transform  $G(f)$  are related by:

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp \{-j2\pi ft\} dt \quad (\text{B-1a})$$

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp \{j2\pi ft\} df \quad (\text{B-1b})$$

a. The following identities are known and useful:

$$\text{i.} \quad \mathcal{F}\{g^*(t)\} \equiv G^*(-f) \quad (\text{B-2a})$$

$$\text{ii.} \quad \mathcal{F}\{g(t-\tau)\} \equiv G(f) \cdot \exp \{-j2\pi f\tau\} \quad (\text{B-2b})$$

$$\text{iii.} \quad \mathcal{F}\{g(t) \exp \{j2\pi\phi t\}\} \equiv G(f-\phi) \quad (\text{B-2c})$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

b. The linear functionals encountered in the text are conveniently handled using Parseval's Theorem<sup>14</sup>

$$\int_{-\infty}^{\infty} u^*(t)v(t)dt = \int_{-\infty}^{\infty} U^*(f)V(f)df . \quad (B-3)$$

c. The linear transformation of one function,  $x(t)$ , into another,  $z(t)$ , by convolution occurs in the form

$$z(t_1) = \int_{-\infty}^{\infty} K(t_1 - t_2)x(t_2)dt . \quad (B-4)$$

Elementary is the knowledge that an equivalent form is

$$z(t_1) = \int_{-\infty}^{\infty} K(f)X(f)\exp\{j2\pi ft_1\}df \quad (B-5)$$

or, more simply,

$$\mathcal{F}\{z(t_1)\} = K(f)X(f) \quad (B-6)$$

d. In several places one must have the value,  $v$ , of a bilinear form having the appearance

$$v = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^*(t_1)K(t_1 - t_2)x(t_2)dt_1 dt_2 . \quad (B-7)$$

When this expression is decomposed into

$$z(t_1) = \int_{-\infty}^{\infty} K(t_1 - t_2)x(t_2)dt_2 \quad (B-8)$$

and

$$v = \int_{-\infty}^{\infty} y^*(t_1)z(t_1)dt_1 \quad (B-9)$$

then application of (B-3) to (B-9), and (B-6) to (B-8), yields

$$v = \int_{-\infty}^{\infty} Y^*(f)K(f)X(f)df . \quad (B-10)$$

2. Optimal Processor for Stationary Interference

The covariance function for statistically stationary interference depends only upon time difference so that, as at Eq. (2.46) of the text,

$$K(t_1, t_2) \equiv K(t_1 - t_2) . \quad (B-11)$$

The covariance function is, moreover, Hermitian. That is

$$K(t_1, t_2) \equiv K^*(t_2, t_1) . \quad (B-12)$$

The power spectral density function  $K(f)$  for the interference is defined by

$$K(f) = \mathcal{F}\{K(\tau)\} , \quad \tau = t_1 - t_2 . \quad (B-13)$$

Using (B-12) one shows that

$$\begin{aligned} K(f) &= \int_{-\infty}^{\infty} K(\tau) \exp \{-j2\pi f\tau\} d\tau \\ &= \int_{-\infty}^{\infty} K^*(-\tau) \exp \{-j2\pi(-f)(-\tau)\} d\tau \\ &= \int_{-\infty}^{\infty} K^*(\tau) \exp \{-j2\pi(-f)\tau\} d\tau \\ &= \left\{ \int_{-\infty}^{\infty} K(\tau) \exp \{-j2\pi(f)\tau\} d\tau \right\}^* \\ &= K^*(f) . \end{aligned} \quad (B-14)$$

Therefore  $K(f)$  is a real function of frequency for Hermitian  $K(\tau)$ .

For stationary interference the optimal processor  $w(t)$  for a signal  $x(t)$  must satisfy

$$\int_{-\infty}^{\infty} K(t_1 - t_2) w(t_2) dt_2 = x(t_1) . \quad (B-15)$$

which stems from (A-9) and (B-11). When Eq. (3-6) is applied to (B-15), one concludes that the Fourier transform of the optimal weight function is given by

$$W(f) = \frac{X(f)}{K(f)} . \quad (B-16)$$

The frequency response function for the optimal processor, as for any linear system, is the complex amplitude of its response to the input  $\exp \{j2\pi ft\}$ . The response to an arbitrary input  $x(t)$  is given at Eq. (2.6) in the form

$$u = \int_{-\infty}^{\infty} w^*(t) x(t) dt . \quad (B-17)$$

The response to  $\exp \{j2\pi ft\}$  is therefore given by

$$u = \int_{-\infty}^{\infty} w^*(t) \exp \{j2\pi ft\} dt \quad (B-18)$$

which may be recognized as the Fourier transform of  $w^*(t)$  evaluated at  $-f$ . Letting  $H(f)$  denote the frequency response function, one may therefore write

$$H(f) = \left[ \mathcal{F} \{w^*(t)\} \right]_{-f} . \quad (B-19)$$

With Eqs. (B-2a), (B-14), and (B-16) this yields

$$H(f) = W^*(f) = \frac{X^*(f)}{K(f)} . \quad (B-20)$$

The cross-ambiguity function  $\xi_{xy}$  for an optimal system with frequency response function given by (B-20) is found by considering its response to an input  $y(t)$  different from  $x(t)$ . In these circumstances the response to be studied

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

is given by

$$\xi_{xy} = \int_{-\infty}^{\infty} w^*(t)y(t)dt \quad . \quad (B-21)$$

Parseval's theorem is used to express the right hand side in the frequency domain, whereupon substitution for  $w^*(f)$  from (B-20) yields

$$\xi_{xy} = \int_{-\infty}^{\infty} \frac{X^*(f)Y(f)}{K(f)} df \quad . \quad (B-22)$$

APPENDIX CTHE KANTOROVICH INEQUALITY

The following is an English language version of the original proof by Kantorovich<sup>†</sup>

"Lemma: The inequality

$$\sum_k \gamma_k x_k^2 \sum_k \gamma_k^{-1} x_k^2 \leq \frac{1}{4} \left[ \sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}} \right]^2 \left( \sum_k x_k^2 \right)^2 \quad (C-1)$$

is valid, where  $m$  and  $M$  are bounds of the numbers  $\gamma_k$

$$0 < m \leq \gamma_k \leq M. \quad (C-2)$$

One may assume that the sums are finite and that

$$\gamma_1 < \gamma_2 < \dots, \quad (C-3)$$

and

$$\sum_k x_k^2 = 1. \quad (C-4)$$

We shall seek the maximum

$$G = \sigma \tilde{\sigma} = \left( \sum_{k=1}^P \gamma_k x_k^2 \right) \left( \sum_{k=1}^P \frac{1}{\gamma_k} x_k^2 \right) \quad (C-5)$$

with the condition

$$\sum_k x_k^2 = 1. \quad (C-6)$$

---

<sup>†</sup> See page 142 of reference 19.



# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

We equate to zero the derivatives of the function

$$F = G - \lambda \left( \sum_{k=1}^P x_k^2 - 1 \right) \quad (C-7)$$

$$\frac{1}{2} \frac{\partial F}{\partial x_s} = \sigma \frac{1}{\gamma_s} x_s + \tilde{\sigma} \gamma_s x_s - \lambda x_s = 0 \quad (C-8)$$

$$\text{i.e.} \quad x_s (\sigma + \tilde{\sigma} \gamma_s^2 - \lambda \gamma_s) = 0 \quad (C-9)$$

The second term in the latter expression is a multinomial of the second degree with respect to  $\gamma_s$ , so that it may become zero for not more than two values of  $s$ ; let these be  $s = k, \ell$ . For the remaining  $s$  it must be that  $x_s = 0$ . But then

$$G_{\max} = \left( \gamma_k x_k^2 + \gamma_\ell x_\ell^2 \right) \left( \frac{1}{\gamma_k} x_k^2 + \frac{1}{\gamma_\ell} x_\ell^2 \right) \quad (C-10)$$

$$= \frac{1}{4} \left[ \sqrt{\frac{\gamma_k}{\gamma_\ell}} + \sqrt{\frac{\gamma_\ell}{\gamma_k}} \right]^2 (x_k^2 + x_\ell^2)^2 - \frac{1}{4} \left[ \sqrt{\frac{\gamma_k}{\gamma_\ell}} - \sqrt{\frac{\gamma_\ell}{\gamma_k}} \right]^2 (x_k^2 - x_\ell^2)^2 \quad (C-11)$$

$$\leq \frac{1}{4} \left[ \sqrt{\frac{\gamma_k}{\gamma_\ell}} + \sqrt{\frac{\gamma_\ell}{\gamma_k}} \right]^2 \leq \frac{1}{4} \left[ \sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}} \right]^2 \quad (C-12)$$

The conditions for equality are readily deduced from the preceding version of the proof. Comparison of equations (C-11) and (C-12) makes it clear that equality is achieved only if

$$1) \quad (x_k^2 + x_\ell^2)^2 + (x_k^2 - x_\ell^2)^2 = 1 \quad (C-13)$$

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

and

$$ii) \quad \gamma_k = m \quad (C-14a)$$

and

$$iii) \quad \gamma_l = M \quad (C-14b)$$

and

$$iv) \quad x_s = 0, \quad s \neq k, s \neq l \quad (C-15)$$

When equation (C-13) is simplified, it reduces to

$$4x_k^2 x_l^2 = 1, \quad (C-16)$$

while equations (C-4) and (C-15) imply

$$x_k^2 + x_l^2 = 1. \quad (C-17)$$

The simultaneous solution of this pair of equations is

$$x_k^2 = x_l^2 = \frac{1}{2}. \quad (C-18)$$

When these values are introduced into equation (C-11) the result is:

$$G_{\max} = \frac{1}{4} \left[ \sqrt{\frac{m}{M}} + \sqrt{\frac{M}{m}} \right]^2. \quad (C-19)$$

An alternative form to the basic inequality (C-1) can be derived if the scalars  $\gamma_k$  are identified with the eigenvalues of a positive definite matrix  $A$ . Then one can write the equivalent inequality

COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

$$(Ax, x) \cdot (A^{-1}x, x) \leq \frac{(m + M)^2}{4mM} (x, x)^2 \quad (C-20)$$

This form of the inequality seems to be the more usually given one<sup>†</sup>. It is certainly the form which, by analogy with equation (4.41) of the text, suggests its application in the present problem.

It may be verified algebraically that the right hand terms of both inequalities (C-1) and (C-20) are identically equal.

---

<sup>†</sup> Forms similar to (C-20) may be seen in Bellman, Ref. 4, p 134; or Beckenbach and Bellman, Ref. 3, P. 70; or Marcus, Ref. 27, p. 11.

APPENDIX D

THE GAUSSIAN FUNCTION

It is well known that the Fourier transform of the Gaussian probability density function with zero mean is given by

$$\mathcal{F}\left[\frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{t^2}{2\sigma^2}\right\}\right] = \exp\left\{-\frac{1}{2}(2\pi f)^2 \sigma^2\right\}. \quad (D-1)$$

If however one defines a new parameter  $W$  by

$$W = \frac{1}{\sqrt{2\pi}\sigma}$$

then one has the preceding relation in the more convenient form used in the text:

$$\mathcal{F}[W \cdot \exp\{-\pi t^2 W^2\}] = \exp\left\{-\pi \frac{f^2}{W^2}\right\}. \quad (D-2)$$

In the latter form  $W$  is a parameter linearly related to the width of the frequency spectrum of the Gaussian time pulse.

Products and quotients of Gaussian functions yield exponential functions with quadratic polynomial exponents. The simplification of such exponents is often facilitated by the identity.

$$\sum_i \alpha_i (t - \beta_i)^2 \equiv \left\{ t - \frac{\sum_i \alpha_i \beta_i}{\sum_i \alpha_i} \right\}^2 \cdot \left( \sum_i \alpha_i \right) + \frac{\sum_i \sum_j \alpha_i \alpha_j \beta_j (\beta_j - \beta_i)}{\sum_i \alpha_i} \quad (D-3)$$

where the right-hand follows upon completion of the square on the left-hand side. If however

$$\sum_i \alpha_i = 0 \quad (D-4)$$

then one can write

$$\sum_i \alpha_i (t - \beta_i)^2 \equiv -2t \left( \sum_i \alpha_i \beta_i \right) + \sum_i \alpha_i \beta_i^2 \quad (D-5)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

## APPENDIX E

### APPROXIMATE INTEGRATION BY HERMITE-GAUSS QUADRATURE

Under certain circumstances, good approximations to the values of definite integrals may be had from the Hermite-Gauss quadrature formula

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) dx \doteq \sum_k \alpha_k^{(M)} f(x_k^{(M)}) \quad (E-1)$$

where the weights  $\alpha_k^{(M)}$  and abscissas  $x_k^{(M)}$  are suitably chosen and independent of the function  $f(x)$ .† Numerical values for the weights and abscissas are given, for  $M$  up to 20, by Salzer, Zucker, and Capuano.<sup>38</sup>

In Chapter X the problem arose of evaluating

$$\left(\frac{S}{I}\right)_{\text{opt}} = 2 \mathcal{E}_s \int_{-\infty}^{\infty} \frac{|M(f; \rho_o, f_o)|^2}{K_c(f) + N_o} df \quad (E-2)$$

where

$$M(f; \rho_o, f_o) = \frac{2^{\frac{1}{4}}}{w^{\frac{1}{2}}} \cdot \exp \left\{ -\pi \frac{(f - f_o)^2}{w^2} \right\} \quad (E-3)$$

$$K_c(f) = 2 \mathcal{E}_c \cdot \frac{1}{w_k} \exp \left\{ -\pi \frac{f^2}{w_k^2} \right\} \quad (E-4)$$

† See Hildebrand<sup>16</sup>, pp. 319-330, for an exposition of the theory of Gaussian quadrature.

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

and 
$$W_k^2 = W_q^2 + \frac{W^2}{2} . \quad (E-5)$$

The required integral in Eq. (E-2) is reduced to the form required for Eq. (E-1) by the following change of variable.

$$x = \frac{\sqrt{2\pi}}{W} (f - f_o) \quad (E-6)$$

When this relation is introduced into equation (E-2), together with the auxiliary function  $G(f)$  defined by

$$G(f) = \frac{1}{K(f) + N_o}, \quad (E-7)$$

the signal-to-interference ratio becomes

$$\left(\frac{S}{I}\right)_{opt} = \frac{2 \mathcal{E}_s}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot G\left(\frac{Wx}{\sqrt{2\pi}} + f_o\right) dx \quad (E-8)$$

or, using Eq. (E-1),

$$\left(\frac{S}{I}\right)_{opt} = 2 \mathcal{E}_s \cdot I_M(f_o, W) \quad (E-9)$$

where 
$$I_M(f_o, W) = \frac{1}{\sqrt{\pi}} \sum_{k=1}^M \alpha_k^{(M)} G\left(\frac{Wx_k^{(M)}}{\sqrt{2\pi}} + f_o\right) \quad (E-10)$$

The "normalized" signal-to-interference ratio parameter  $\mu_o$  is computed according to

$$\mu_o = \frac{2 \mathcal{E}_c}{W_q} \cdot I_M(f_o, W) \quad (E-11)$$

# COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

It is therefore clear that by the foregoing procedure one may choose val for  $f_o$  and  $W$  and compute approximations to the corresponding values of  $\mu_o$  or  $(\frac{S}{I})_{opt}$ , once the value of  $M$  has been decided upon. Computations of  $I_M(f_o, W)$  for different  $M$ , and the same  $f_o$  and  $W$ , will yield different approximate values for  $\mu_o$  and  $(\frac{S}{I})_{opt}$ . In general one expects that the larger the value of  $M$  is (leading to more terms in the approximating sum) the better will be the approximation. A small  $M$ , however, would be preferred for reasons of economy in generating the data.

For the data presented in the text  $I_M(f_o, W)$  has been taken to be the value given by the smallest  $M$  for which

$$\left| \frac{I_M(f_o, W) - I_{M-1}(f_o, W)}{I_M(f_o, W)} \right| < \epsilon \quad (E-12)$$

for a specified tolerance  $\epsilon$ . For the values of  $f_o$ ,  $W$ , and  $R_c$  which appear in the text, the inequality (E-12) was satisfied for  $M \leq 20$  with  $\epsilon$  equal to 0.01.

The actual numerical values of the various parameters appearing in the calculations were chosen from among

$$\begin{aligned} \mathcal{E}_s &= 100 \\ \mathcal{E}_c &= 5000 \\ v_q &= 100 \\ \left(\frac{f_o}{W}\right)^2 &= 0, 1, 2, 3, 4, 5, 10, 10^2, 10^3, 10^4 \\ N_o &= 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10, 100 \\ \left(\frac{W}{W_q}\right) &= .1, .316, 1, 3.16, 10, 31.6, 100 \end{aligned}$$

with occasional other values used to fill in data required for better visualization of rapidly varying data.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

### REFERENCES

1. Barlow, J. E., "Doppler Radar," IRE Proceedings, Vol. 37, April 1949, pp. 340-355.
2. Barton, D. K., "Radar System Analysis," Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1964.
3. Beckenbach, E. F., and Bellman, R., "Inequalities," Vol. 30 of "Ergebnisse der Mathematik und Ihrer Grenzgebiete," Springer-Verlag, Berlin, 1961.
4. Bellman, R., "Introduction to Matrix Analysis," McGraw-Hill Co., New York, 1960.
5. Bernstein, R. I., "An Analysis of Angular Accuracy in Search Radar," Columbia University Electronics Research Laboratories Technical Report, T-1/128, May 1955.
6. Bode, H. W., and Shannon, C. E., "A Simplified Derivation of Linear Least-Square Smoothing and Prediction Theory," Proc. IRE, Vol. 38, April 1950, pp. 417-425.
7. Booton, R. C., "An Optimization Theory for Time-Varying Linear Systems with Nonstationary Statistical Inputs," Proc. IRE, Vol. 40, August 1952, pp. 977-981.
8. Courant, R., and Hilbert, D., "Methods of Mathematical Physics, Vol. 1," Interscience Publishers Inc., New York, 1953.
9. Churchill, R. V., "Operational Mathematics," Second edition, McGraw-Hill Co., New York, 1958.
10. Dickey, F. R., "Theoretical Performance of Airborne Moving Target Indicators," IRE Transactions on Aeronautical and Navigational Electronics, June 1953, pp. 12-23.
11. Dugundji, J., "Envelopes and Pre-Envelopes of Real Waveforms," IRE Transactions on Information Theory, March 1958, pp. 53-57.



## COLUMBIA UNIVERSITY--ELECTRONICS RESEARCH LABORATORIES

12. Dwork, B. M., "Detection of a Pulse Superimposed on Fluctuation Noise," Proc. IRE, Vol. 38, July 1950, pp. 771-774.
13. Fowle, E. N., Kelly, E. J., and Sheehan, J. A., "Radar System Performance in a Dense-Target Environment," IRE Convention Record - Part IV, 1961, pp. 136-145.
14. George, T. S., "Fluctuations of Ground Clutter Return in Airborne Radar Equipment," Proceedings of the I.E.E., Vol. 99, Part IV, No. 2, April 1952, pp. 92-98.
15. Helstrom, C. W., "Statistical Theory of Signal Detection," Pergamon Press, New York, 1960.
16. Hildebrand, F. B., "Introduction to Numerical Analysis," McGraw-Hill Co., New York, 1956.
17. James, H. M., Nichols, N. B., Phillips, R. S., "Theory of Servomechanisms," MIT Radiation Laboratory Series, Vol. 25, McGraw-Hill, New York, 1947.
18. Kailath, T., "Correlation Detection of Signals Perturbed by a Random Channel," IRE Transactions on Information Theory, June 1960, pp. 361-366.
19. Kantorovich, L. V., "Funktsionalnii Analiz i Prikladnaya Matematika," Uspehi Matematicheskikh Nauk, Vol. 3, November - December 1948, pp. 89-185.
20. Kerr, D. E., "Propagation of Short Radio Waves," M.I.T. Radiation Laboratory Series, Vol. 13, McGraw-Hill Co., New York, 1951.
21. Key, E. L., Fowle, E. N., and Haggarty, R. D., "A Method of Designing Signals of Large Time-Bandwidth Product," IRE Convention Record - Part IV, 1961, pp. 146-154.
22. Korn, G. A., and Korn, T. M., "Mathematical Handbook for Scientists and Engineers," McGraw-Hill Co., 1961.
23. Krinitz, A., "A Radar Theory Applicable to Dense Scatterer Distributions," Report ESL-R-131, M.I.T. Electronic Systems Laboratory, Cambridge, Mass., January 1962.
24. Lawson, J. L., and Uhlenbeck, G. E., "Threshold Signals," M.I.T. Radiation Laboratory Series, Vol. 24, McGraw-Hill, New York, 1950.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

25. Lerner, R. M., "A Matched Filter Detection System for Complicated Doppler Shifted Signals," IRE Transactions on Information Theory, June 1960, pp. 373-385.
26. Lovitt, W. V., "Linear Integral Equations," Dover Publications, New York, 1950.
27. Marcus, M., "Basic Theorems in Matrix Theory," National Bureau of Standards Applied Mathematics Series, No. 57, U. S. Department of Commerce, 1960.
28. Margenau, H., and Murphy, G. M., "The Mathematics of Physics and Chemistry, Second Edition," Van Nostrand, New York, 1956.
29. Miller, K. S., and Zadeh, L. A., "Solution of an Integral Equation Occurring in the Theories of Prediction and Detection," IRE Transactions on Information Theory, June 1956, pp. 72-75.
30. North, D. O., "An Analysis of the Factors Which Determine Signal/Noise Discrimination in Pulsed-Carrier Systems" I.E.E.E. Proceedings, July 1963, pp. 1016-1027. (Publication of RCA Laboratories Report PTR-6 C, dated June 25, 1943.)
31. Pinney, E., "Ordinary Difference-Differential Equations," University of California Press, Berkeley and Los Angeles, California, 1958.
32. Price, R., "The Detection of Signals Perturbed by Scatter and Noise," IRE Transactions on Information Theory, September 1954, pp. 163-169.
33. Price, R., and Green, Jr., P. E., "Signal Processing in Radar Astronomy-Communication via Fluctuating Multipath Media," MIT Lincoln Laboratory Technical Report No. 234, October 6, 1960.
34. Price, R., and Hofstetter, E. M., "Bounds on the Volume and Height Distributions of the Ambiguity Function," I.E.E.E. Transactions on Information Theory, April 1965, pp. 207-214.
35. Ridenour, L. N., "Radar System Engineering," MIT Radiation Laboratory Series, Vol. 1, McGraw-Hill Co., New York, 1947.

36. Riesz, F., and Sz.-Nagy, B., "Functional Analysis," Frederick Ungar Publishing Co., New York, 1955.
37. Rihaczek, A. W., "Radar Signal Design for Target Resolution," Proceedings of the I.E.E.E., February 1965, pp. 116-128.
38. Salzer, H. E., Zucker, R., and Capuano, R., "Table of the Zeroes and Weight Factors of the First Twenty Hermite Polynomials," Journal of Research of the N. B. S., Vol. 48, No. 2, February 1952, pp. 111-116.
39. Skolnik, M., "Introduction to Radar Systems," McGraw-Hill, New York, 1962.
40. Stewart, J. L., and Westerfield, E. C., "A Theory of Active Sonar Detection, IRE Proceedings, May 1959, pp. 872-881.
41. Stutt, C. A., "A Note on Invariant Relations for Ambiguity and Distance Function," IRE Transactions on Information Theory, December 1959, pp. 164-167.
42. Tack, D. H., Ziemer, R. E., and Lambert, R. F., "Coherent Extraction of Signals from Turbulent Noise," I.E.E.E. Transactions on Information Theory, January 1964, pp. 78-88.
43. Turin, G. L., "An Introduction to Matched Filters," IRE Transactions on Information Theory, June 1960, pp. 311-329.
44. Urkowitz, H., "Analysis and Synthesis of Delay Line Periodic Filters," IRE Transactions on Circuit Theory, June 1954, pp. 41-53.
45. Urkowitz, H., "Filters for Detection of Small Radar Signals in Clutter," J. Appl. Phys., Vol. 24, No. 8, August 1953, pp. 1024-1031.
46. Westerfield, E. C., Prager, R. H., and Stewart, J. L., "Processing Gains Against Reverberation (Clutter) Using Matched Filters," IRE Transactions on Information Theory, June 1960, pp. 342-348.

## COLUMBIA UNIVERSITY—ELECTRONICS RESEARCH LABORATORIES

47. White, W. D., and Ruvin, A. E., "Recent Advances in the Synthesis of Comb Filters," IRE National Convention Record - Part 2, 1957, pp. 186-199.
48. Wiener, N., "Extrapolation, Interpolation, and Smoothing of Stationary Time Series," John Wiley and Sons, New York, 1949.
49. Woodward, P. M., "Probability and Information Theory with Applications to Radar," Pergamon Press, New York, 1953.
50. Zadeh, L. A., and Ragazzini, J. R., "An Extension of Wiener's Theory of Prediction," J. Appl. Phys., July 1950, pp. 645-655.
51. Zadeh, L. A., and Ragazzini, J. R., "Optimum Filters for the Detection of Signals in Noise," IRE Proceedings, October 1952, pp. 1223-1231.